Valuation of International Corporate Debt.

Ignacio Warnes Gabriel Acosta

Abstract

Many large corporations split their debt between two (or more) currencies. Therefore, their default risk and also their yield rates are influenced by the exchange rate process between those currencies. This paper presents a model of corporate debt valuation that captures this influence, effectively extending the default risk analysis of Merton (1974) to a two currency world. We obtain the corporate yield in each of the currencies, and show the credit risk effect of foreign currency exposure on claims value.

We develop a model to value the debt issued by a corporation, in the presence of default risk and exchange rate risk. We consider the company's production decision as given, and assume that it takes place in one country. The corporation has the option to issue debt either in the domestic capital market, or in the foreign capital market, or choose a mix of both. The valuation of risky corporate claims in a single currency world, has produced a large literature, since Merton (1974) defined the event of bankruptcy, stating that it occurs when the firm's assets value falls below its liabilities. In Merton (1974), the company's debt value is affected by its default probability, which stems from the states of the world in which the company's value is smaller than its outstanding obligations. Black and Cox (1976) modify Merton's definition of bankruptcy, stating that it occurs when the firm's asset value reaches a pre- specified (fixed) boundary. Longstaff and Schwartz (1995) follow Black and Cox's definition of default. Moreover, they incorporate interest rate risk, by defining the stochastic (mean reverting) process that governs default free interest rates. Madan and Unal

(1998) propose a model where default does not result from the interaction of observable variables, but, the probability of default evolves over time, being always positive. Unlike Merton, they do not define the state of the world at which default occurs, and treat default as the result of a pure jump process. Other related papers include Ramaswamy, Kim and Sundaresan (1986), Jarrow, Lando, Turnbull (1997), Madan and Unal (2000), and also Jarrow and Turnbull (1995).

To define the default occurrence we follow the explicit variables and endogenously defined state of the world approach. This continues the view that Merton (1974), Black and Cox (1976) and Longstaff and Schwartz (1995) apply to domestic corporate debt valuation. Since our aim is to concentrate on two risk factors, asset value risk and exchange rate risk, we choose to follow Merton's original model, where interest rates were constant. Moreover, we generalize Merton's model, incorporating a second risk factor, given by the exchange rate movements that affect the value of the debt denominated in the foreign currency. As in Adler and Dumas (1983), we will assume that the default free returns (interest rates) in the foreign currency can be modelled as a log-normal random variable.

We understand that this is the first analytic model where the valuation of risky corporate debt issued in two different currencies is obtained, taking into account both default risk and exchange rate risk. Some related issues have been studied in the international finance literature. Mehra (1978) analyzes the effects of exchange rate fluctuations in an international CAPM context. In that setting, he showed that the Modigliani - Miller proposition still holds. Also, Adler and Dumas (1983) (Sections 4 and 8) analyze the assumptions on world capital market perfectness under which the Modigliani - Miller theorem holds in an international setting. On the other hand, Bakshi and Chen (1997), using a continuous times Lucas (1982) two countries economy study the valuation of foreign exchange claims. Their aim is to value options and other derivatives written on foreign currencies. Nevertheless, they do not address the valuation of corporate liabilities subject to default risk. Other related papers include Mello, Parsons and Triantis (1993), and also Amin and Jarrow (1991).

We will write V and X to denote the value of the company in the domestic currency, and the exchange rate respectively (i.e. the price in domestic currency of one unit of foreign currency) [1].

We will assume that the corporation promises to pay, at maturity time T, one part of its debt in domestic currency and another portion in foreign currency, say B_d and B_f respectively. Hence the total obligations of the firm, (measured in local currency) at maturity date T will be $B_f X_T + B_d$ for a given value of X_T. This amount coincides with the debt value at T under the assumption of no default [2].

Following Merton (1974) we will establish the differential equation that rules the evolution of the debt value F assuming that $F = F(V, X, t)$ and under the assumptions on V and X given by (1). This is accomplished in Section 1.

In Section 2 we show that an explicit solution can be given if the debt is issued completely in foreign currency (i.e. $B_d = 0$). The case $B_f = 0$ also has an explicit solution, as stated in Merton´s model. Moreover, we show that, when both B_d and B_f are positive, an explicit solution is obtained provided that V and X are not correlated. This result is given in Section 3. In Section 4 we isolate the value of each part of the debt, i.e. we give an explicit solution F_d and F_f for the debt issued in domestic and foreign currencies respectively. The aim of this section is to show the reference yield rates that should hold in each of the two currencies. We assume a proportional payment at maturity in case of default ("pars creditorum conditio"). We mean that if at maturity the value V of the company is below the value of the debt then the section issued in domestic currency will pay $V\frac{B_d}{X_T B_f + Bd}$ and the foreign currency section will grant $V\frac{X_T B_f}{X_T B_f + Bd}$.

1 The model

The firm's value V and the exchange rate X are assumed to be stochastic variables. More precisely we will model them as follows

$$\begin{cases} dV = c_V V dt + \sigma V dz \\ dX = c_X X dt + \sigma^* X dz^* \end{cases} \tag{1}$$

where z and z^* stand for Wiener processes. Since V and X are functions of time we will sometimes use V_t and X_t to denote them (however when there is no risk of confusion we will drop the t.) We will assume that in both markets, there are default-free, zero beta rates. They will be denoted r_d and r_f respectively, the riskless rates in domestic and foreign currencies.

We call F to the value as of time t of the total contractually promised payments B_f and B_d . This value F is measured in domestic currency. Observe that V/X is the company value measured in foreign currency, and F/X is the total debt contract value measured in foreign currency.

We assume that the price of the debt F in domestic currency depends only on the corporate value, the exchange rate, and the time t, i.e.

$$F = F(V, X, t). \tag{2}$$

Applying Ito's Lemma, we obtain from (2)

$$dF = F_V dV + F_X dX + F_t dt + \frac{1}{2}(F_{VV} dV^2 + 2F_{XV} dV dX + F_{XX} dX^2) \quad (3)$$

Following Merton (1974) we consider a suitable portfolio $\Pi = F - \Delta_X X - \Delta_V V$, including Δ_X units in the default-free investment denominated in foreign currency and Δ_V units of the underlying stock of the corporation. The instant change during dt of such a portfolio can be written as

$$d\Pi = dF - \Delta_X dX - \Delta_V dV - d\Delta_X X \quad (4)$$

Let us note that $d\Delta_X$ is not identically zero since, during the period dt, the amount Δ_X is invested at a riskless rate r_f an so it yields in domestic currency $r_f \Delta_X X dt$ hence

$$d\Pi = dF - \Delta_X dX - \Delta_V dV - r_f \Delta_X X dt \quad (5)$$

From this equation together with (1) and (3) we find that the stochastic terms of $d\Pi$ can be eliminated by means of an appropriate selection of Δ_X and Δ_V, more precisely, by taking $\Delta_X = \frac{\partial F}{\partial X} = F_X$ and $\Delta_V = \frac{\partial F}{\partial V} = F_V$ we obtain

$$d\Pi = (F_t + \frac{1}{2}(\sigma^2 V^2 F_{VV} + 2\rho X V F_{XV} + \sigma^{*2} X^2 F_{XX}) - r_f X F_X) dt \quad (6)$$

With this choice of Δ_X and Δ_V, the portfolio is no longer risky in the interval dt, and thus we can evaluate its price evolution by means of the risk free rate in domestic currency r_d. We can write:

$$d\Pi = r_d (F - \frac{\partial F}{\partial X} X - \frac{\partial F}{\partial V} V) \quad (7)$$

and from (6) and (7) we obtain the differential equation for the debt

$$
\begin{aligned}
F_t + \frac{1}{2}\sigma^2 V^2 F_{VV} + \rho\sigma\sigma^* XVF_{XV} + \frac{1}{2}\sigma^{*2} X^2 F_{XX} \quad & + \\
r_d V F_V + (r_d - r_f) X F_X - r_d F \quad & = \quad 0
\end{aligned}
\tag{8}
$$

For this equation we have to impose both initial and boundary conditions. The claim value at maturity time T is the minimum between the value of the corporation and $B_f X + B_d$, therefore we can write

$$
F(V, X, T) = \min(V, B_f X + B_d) \tag{9}
$$

Now, on the one hand, the debt becomes worthless when $V = 0$, and on the other when $X = 0$ the value of the debt can be computed by means of the classical Merton approach. In fact, if $X = 0$ the foreign debt could be repurchased at no cost and the company would pay only the domestic part B_d. Hence

$$
\begin{cases}
F(0, X, t) = 0 \\
F(V, 0, t) = W(V, t)
\end{cases}
\tag{10}
$$

where $W(X, t)$ is the Merton's solution for the debt placed completely in domestic currency with the promise to pay B_d at the maturity time T.

Conditions (10) can be summarized as

$$
F(V, X, t) \leq \min(V, B_f X + B_d) \tag{11}
$$

The differential equation (8) under the initial and boundary conditions (9) and (11), respectively could be solved numerically by difference methods. However, as we will see, an explicit solution can be obtained in several cases. More precisely, under the hypothesis that r_f and r_d are constant we will show that an explicit solution can be obtained in all the following cases:

1. $B_d = 0$ or

2. $B_f = 0$ (this is indeed the Merton case) or

3. The mixed currencies case (i.e. $B_d \neq 0$ and $B_f \neq 0$), with $\rho = 0$

2 All debt in foreign currency

In this section we will assume that the company promises to pay a given amount B_f in foreign currency, and that there is no debt issued in the domestic money.

From (9), and (11), we see that in this case the initial and boundary conditions take the form

$$F(V, X, T) = \min(V, B_f X) \tag{12}$$

and

$$\begin{cases} F(0, X, t) = 0 \\ F(V, 0, t) = 0 \end{cases} \tag{13}$$

Using now the standard technique we introduce:

$$f = V - F \tag{14}$$

and an straightforward computation shows that f complies with the equation (2) under the following conditions

$$f(V, X, T) = \max(0, V - B_f X) \tag{15}$$

$$0 \leq f \leq V \tag{16}$$

We will show that a further change of variables $Z = \frac{V}{X}$ recovers a call option equation, similarly to what Merton found for the case $B_f = 0$. Defining the auxiliary function $f^*(Z,t) = \frac{f(V,X,t)}{X}$. One can easily check that f^* verifies:

$$f_t^* + \frac{1}{2}(\sigma^2 - 2\rho\sigma\sigma^* + \sigma^{*2}) f_{ZZ}^* + r_f f_Z^* - r_f f = 0 \tag{17}$$

and

$$f^*(Z,T) = \max(0, Z - B_f) \tag{18}$$

$$0 \leq f^* \leq Z. \tag{19}$$

An explicit expression for f^* , and hence for f, can be given from well known methods. In fact, calling $\delta^2 = \sigma^2 - 2\rho\sigma\sigma^* + \sigma^{*2}$ we get

$$f^*(Z,t) = ZN(d_1) - B_f \exp(-r_f(T-t))N(d_2) \tag{20}$$

where

$$d_1 = \frac{\ln(\frac{Z}{B_f}) + (r_f + \frac{1}{2}\delta^2)(T-t)}{\delta\sqrt{T-t}} \tag{21}$$

$$d_2 = \frac{\ln(\frac{Z}{B_f}) + (r_f - \frac{1}{2}\delta^2)(T-t)}{\delta\sqrt{T-t}} \tag{22}$$

$$N(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{1}{2}s^2} ds \tag{23}$$

Equations (17), (18), (19) represent a call option with exercise price B_fLet us now derive F. Since

$$F(V,X,t) = V - B_f X f^*(\frac{V}{X}, t) \qquad (24)$$

we obtain from (20)

$$F(V,X,t) = V(1 - N(d_1)) + B_f X \exp(-r_f(T-t))N(d_2) \qquad (25)$$

Calling $\tau = T - t$ the time to maturity, we can obtain an expression for the yield to maturity of the risky debt. Following Merton (1974) we compute this yield, and the other yields through the rest of the paper based on the assumption that the company does not default. In order to do so we need to express the value of the debt and the promised payment at maturity in the same currency. We do so as follows:

$$Y_1 = -\frac{\ln(\frac{F(V,X,T-\tau)}{B_f X})}{\tau}$$

where in the numerator the quotient inside the logarithm can be written as

$$\frac{\frac{F(V,X,T-\tau)}{X}}{B_f}$$

i.e. the ratio of the debt contract (measured in foreign currency) to the value of the principal in the same currency. So, Y_1 is the yield obtained in foreign currency.

3 Domestic and Foreign currency combined

In this section we will write an explicit solution for the debt pricing under the following conditions: The company promises to pay at some maturity time T , a total amount of B_d and B_f in domestic and foreign currency respectively, with $B_d > 0$ and $B_f > 0$: The company issues debt in both currencies. For simplicity the correlation between the value of the firm and the exchange rate is assumed to be worth zero, $\rho = 0$.

Hence our objective is to solve equation (8) under the conditions (11) and (9).

By means of the following changes of variables

$$\overline{V} \to \frac{\sqrt{2}}{\sigma}\ln(V)\text{and } \overline{X} \to \frac{\sqrt{2}}{\sigma^*}\ln(X)$$

we can transform (8) into a constant coefficient equation. For simplicity we will drop the bars in the new variables, i.e. V and X stands for the new variables, although when we find the solution we should recall this fact.

From (8) and by means of the change of variables we arrive to

$$F_t + F_{VV} + F_{XX} + \beta F_V + \gamma F_X - r_d F = 0 \tag{26}$$

where

$$\beta = \frac{\sqrt{2}}{\sigma}(r_d - \frac{\sigma^2}{2}) \text{ and } \gamma = \frac{\sqrt{2}}{\sigma^*}(r_d - r_f - \frac{\sigma^{*2}}{2})$$

the conditions (11) and (9) become respectively

$$F(V, X, t) \leq \min(e^{\frac{\sigma V}{\sqrt{2}}}, B_f e^{\frac{\sigma^* X}{\sqrt{2}}} + B_d) \tag{27}$$

$$F(V, X, T) = \min(e^{\frac{\sigma V}{\sqrt{2}}}, B_f e^{\frac{\sigma^* X}{\sqrt{2}}} + B_d) \tag{28}$$

Introducing now a new function $G(V, X, t) = e^{-at-bV-cX}F(V, X, t)$ and selecting adequately a, b and c the function G satisfies the heat equation in the whole plane.

In fact, if

$$a = r_d + \frac{\beta^2}{2} + \frac{\gamma^2}{2} \quad b = -\frac{\beta}{2} \quad c = -\frac{\gamma}{2}$$

we obtain

$$G_t + G_{VV} + G_{XX} = 0$$

with

$$G(V, X, t) \leq e^{-at-bV-cX} \min(e^{\frac{\sigma V}{\sqrt{2}}}, B_f e^{\frac{\sigma^* X}{\sqrt{2}}} + B_d)$$

$$G(V, X, T) = e^{-at-bV-cX} \min(e^{\frac{\sigma V}{\sqrt{2}}}, B_f e^{\frac{\sigma^* X}{\sqrt{2}}} + B_d)$$

which can be solved by means of the convolution of the initial datum with the fundamental solution

$$\Phi(\xi, \eta, t) = \frac{1}{4\pi(T-t)} e^{-\frac{\xi^2+\eta^2}{4(T-t)}}$$

indeed

$$G(V, X, t) = \int\int_{R^2} \Phi(V-\xi, X-\eta) e^{-at-b\xi-c\eta} \min(e^{\xi}, B_f e^{\eta} + B_d) d\xi d\eta \quad (29)$$

and the solution of the original equation (8) in terms of the new variables can be obtained from here

$$F(V, X, t) = e^{at+bV+cX} G(V, X, t)$$

which in terms of the original variables reads

$$F(V,X,t) = e^{at} V^{\frac{\sqrt{2}}{\sigma} b} X^{\frac{\sqrt{2}}{\sigma^*} c} G\left(\frac{\sigma}{\sqrt{2}} \ln(V), \frac{\sigma^*}{\sqrt{2}} \ln(X), t\right) \tag{30}$$

Let us note that even when this expression represents an explicit solution of our problem it is not posed in a friendly form. Clearly the double integral involved in (29) can be computed numerically by means of standard methods and using the nice decaying properties of the fundamental solution, however we will show a closed form solution for (30) involving only an ordinary integral and hence in a form that is easier to handle. Since the computations are long but not difficult we omit the details.

From (29) and (30), one can check that $F(V,X,t)$ can be written by means of the following terms

$$F(V,X,t) = VI_1 + e^{-(T-t)r_f} B_f X I_2 + e^{-(T-t)r_d} B_d I_3 \tag{31}$$

with

$$I_1 = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (1 - N(I_{11})) e^{-\frac{1}{2}s^2} ds \tag{32}$$

$$I_2 = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} N(I_{22})\, e^{-\frac{1}{2}s^2} ds \tag{33}$$

$$I_3 = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} N(I_{33})\, e^{-\frac{1}{2}s^2} ds \tag{34}$$

where

$$I_{11} = \frac{\ln\left(\frac{V}{B_f X e^{s\sigma^*\sqrt{T-t}-\frac{\sigma^{*2}}{2}(T-t)+(r_d-r_f)(T-t)}+B_d}\right)+(r_d+\frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}} \quad (35)$$

$$I_{22} = \frac{\ln\left(\frac{V}{B_f X e^{s\sigma^*\sqrt{T-t}+\frac{\sigma^{*2}}{2}(T-t)+(r_d-r_f)(T-t)}+B_d}\right)+(r_d-\frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}} \quad (36)$$

$$I_{33} = \frac{\ln\left(\frac{V}{B_f X e^{s\sigma^*\sqrt{T-t}-\frac{\sigma^{*2}}{2}(T-t)+(r_d-r_f)(T-t)}+B_d}\right)+(r_d-\frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}} \quad (37)$$

In the appendix, we will check the solution (31) that we have just obtained, verifying that it holds in the following three limit cases: when the debt is completely issued in foreign currency, when it is totally denominated in domestic currency and, finally, when the exchange rate volatility is equal to zero. We choose those three cases not only for its singular importance, but also because in those three cases the solutions values are either well known or can easily be reduced to a well known equation.

4 Yield rates in the two currencies

In the preceding section we have found the value of the mixed currencies debt contract, hence we may ask for an expression of the yield. However a problem naturally arises: the debt is measured totally in domestic currency but the payment at maturity is mixed in domestic and foreign currencies and stochastic and hence its actual value is not known until maturity. There are three different yield concepts that we can analyze: The total contract yield, the yield of the domestic currency section and the yield of the foreign currency part and, in this section, all three concepts are computed assuming the market expectation on the foreign exchange rate are given by the market value of the forward rate,

$$E(X_T) = X_T^F.$$

The previous condition states that the expected return from investing in risk free instruments in any of the two curencies has to be same. This is a standard condition under risk neutral valuation. We will first introduce the concept of forward exchange rate X_T^F

$$X_T^F = X_{t_o} e^{\int_{t_0}^{T} (r_d(s) - r_f(s)) ds}$$

and from the fixed risk-free rates rate that we are assuming this implies

$$X_T^F = X_{t_o} e^{(r_d - r_f)(t - t_0)}.$$

At some time to maturity $\tau = T - t$ the firms promises to pay, if does not default, a total amount of

$$E(B_f X + Bd) = B_f X_T^F + Bd = BfX \exp(c_X \tau) + Bd$$

hence we can write for the total yield

$$Y_{Total} = -\frac{\log(\frac{F(X,V,T-\tau)}{BfX \exp(c_X \tau) + Bd})}{\tau} \tag{38}$$

where F is given by (31).

We will also isolate each part of the debt along the time. In this case the payment at maturity of each section, under the assumption of no default, will no longer be stochastic. Suppose, then, that we call $F_f(V,X,t)$ and $F_d(V,X,t)$ to the debt value in foreign and domestic part respectively at time t, can we get for each part an expression similar to that given by (31)?. In fact identical arguments used for $F(V,X,t)$, shows that $F_f(V,X,t)$ and $F_d(V,X,t)$ should verify (8). The main difference appears in the way that the contract defines the payment in case of default.

A reasonable assumption could be the usual "pars creditorum conditio ", applicable when there is no predefined seniority among creditors. The "pars creditorum conditio" establishes that the company´s remaining assets, in case of bankruptcy, should be divided among its creditors proportionally to the value of each creditor´s claim.

$$F_f(V,X,T) = \begin{cases} XB_f & if \quad XB_f + B_d \leq V \\ XB_f \frac{V}{XB_f + B_d} & if \quad XB_f + B_d > V \end{cases}$$

$$F_d(V,X,T) = \begin{cases} B_d & if \quad XB_f + B_d \leq V \\ B_d \frac{V}{XB_f + B_d} & if \quad XB_f + B_d > V \end{cases}$$

in this case, the boundary conditions are

$$F_f(0,X,t) = 0 \text{ and } F_f(V,0,t) = 0$$

$$F_d(0,X,t) = 0 \text{ and } F_d(V,0,t) = F(V,0,t).$$

One can easily prove using the linearity of (8) and summing up the final data that

$$F(V,X,t) = F_f(V,X,t) + F_d(V,X,t) \tag{39}$$

however from this identity alone we can not establish the value of F_d nor F_d. Despite this fact we can integrate any of them just following the same steps used to look for F.

Indeed an straightforward computation yields

$$F_f(0,X,t) = VL_f + e^{-r_f(T-t)}XB_fI_2$$

$$F_d(0,X,t) = VL_d + e^{-r_d(T-t)}B_dI_3$$

where I_2 and I_3 are defined in (33) (34), and

$$L_f = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{B_fXe^{(s-c\sqrt{2}\sqrt{T-t})\sigma^*\sqrt{T-t}}(1-N(I_{11}))}{(B_fXe^{(s-c\sqrt{2}\sqrt{T-t})\sigma^*\sqrt{T-t}}+Bd)}e^{-\frac{1}{2}s^2ds}$$

$$L_d = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{B_d(1-N(I_{11}))}{(B_fXe^{(s-c\sqrt{2}\sqrt{T-t})\sigma^*\sqrt{T-t}}+B_d)}e^{-\frac{1}{2}s^2ds}$$

Let us notice that one can easily get from (31) and (39) that

$$L_d + L_f = I_1.$$

Since we have the value of each part of the debt we can find the yields that correspond to the debt issued in domestic and foreign currencies. Indeed the yield of the domestic currency issue is

$$Y_d = -\frac{\ln(\frac{F_d(V,X,t)}{B_d})}{T-t} \tag{40}$$

and the yield of the foreing currency issue is

$$Y_f = -\frac{\ln(\frac{F_f(V,X,t)}{XB_f})}{T-t} \tag{41}$$

Figure 1 shows the effect that the changes in both the volatility of firm value σ^2 and the volatility of exchange rate ${\sigma^*}^2$ have on the total yield risk premium.

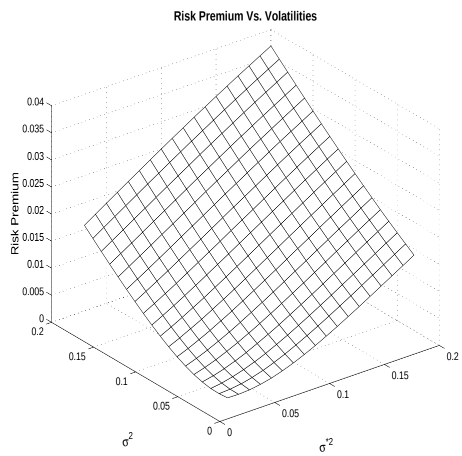

Figure 1

Figure 2 below shows the change in the total risk yield premium with the variation in firm's value volatility. The exchange rate's volatility is assumed constant in this figure.

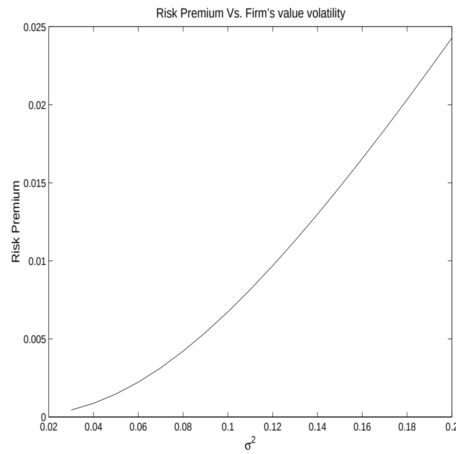

Figure 2

Figure 3 shows the change in the domestic issue yield premium with the variation in firm's value volatility. The exchange rate's volatility is assumed constant in this figure.

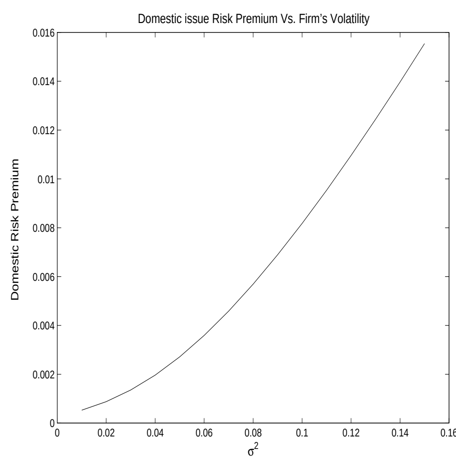

Figure 3

5 Comparative Statics

In this section, we present some results derived from the valuation equation (31). We observe that:

$F(V, X, t) = VI_1 + e^{-(T-t)r_f} B_f X I_2 + e^{-(T-t)r_d} B_d I_3$

shows that the value of the international corporate debt issue is equal to the value of an equivalent portfolio. The equivalent portfolio consists of the sum of I_1company shares, plus the amount $e^{-(T-t)r_d} B_d I_3$ invested in the domestic risk-free security, and another amount $e^{-(T-t)r_f} B_f X I_2$ invested in the foreign currency denominated riskfree security.

We analyze here some properties of the relative values between these three components of the equivalent portfolio. All of these properties are proved in the Appendix.

Property 1: If $\sigma^* \sqrt{T-t} \geq 3 => \frac{d(e^{-(T-t)r_d} B_d I_3 / e^{-(T-t)r_f} B_f X I_2)}{d\sigma^*} > 0$

When foreign exchange volatility is large and (or) time to maturity is significant, we obtain that an incease in foreign exchange volatility implies that the relative value of the domestic riskfree investment increases vis a vis the value of the foreign currency riskfree investment.

Property 2: If $\sigma^* \sqrt{T-t} \geq 3 => \frac{d(e^{-(T-t)r_d} B_d I_3 / V I_1)}{d\sigma^*} > 0$

This says that an increase in foreign exchange volatilty implies that the relative vaue of the domestic riskfree investment increases w.r.t. the relative value of equity, when foreign exchange volatility is large and (or) time to maturity is significant.

In Merton's model, we know that debt can be interpreted as a call option on equity. Therefore, more price volatility for the company stock means more value for the option, implying that we obtain a larger value for the debt. In our case,an increase in foreign exchange volatility implies that the strike price of the corporate option to buy back its own equity becomes more volatile. In this context (for σ^* or T-t reasonably larges) the foreign exchange volatility increment results in a larger value of the domestic component of debt vis a vis the company´s equity value. It is important to remark that the intuition here is different from the one at Merton (domestic) model, since here the increase in foreign exchange volatility is affecting the option ´strike price rather than the equity´s volatility.

6 Explicit valuation for European derivatives dependent on two lognormal processes.

We will show that the pricing formula (31) found in section 3 applies indeed into a more general situation. Let us call F to the price of a call option, that is written on a security V, that follows a log normal process. Assume that this call exercise price is a linear function of another security price, X, that also follows a lognormal process.

Therefore we have the problem of pricing a call, whose price depends on two different lognormal processes, X and V:

$F = F(V, X, t)$

with

$F(V_T, X_T, T) = Max(V_T, A + BX_T)$

Under the assumption that $\rho(V, X) = 0$, the equation (31) shows the closed form solution for this European call, where the value A should substitute B_d and the B should replace B_f.

This pricing formula generalizes Black and Scholes in two senses: in the first place, the derivative is now a function of two log normal processes, rather than one- as is the case in the Black and Scholes formula. In the second place, the strike price itself is permitted, here, to be a linear function of one of the two lognormal processes (rather than a constant).

Notice that in this derivation, the asset X_t that enters the exercise price is permitted to pay a dividend proportional to the value of the asset, whose rate is given by r_f.

7 CONCLUSION

We have found a closed form valuation equation for corporate debt, when the debt value depends on two lognormal risk factors, company value and exchange rate. This determination of a closed form solution for Merton's problem with two risk factors is one of this papers main contribution. The valuation equation provides the yield rates that a corporation should pay in each of the two currencies in which it issues debt, as a function of the company's value and the exchange rate process.

As future research, we believe it important to study wether this closed form solution can be extended to the case of non-zero correlation between the two lognormal risk factors. We also find relevant to extend our results to the case of stochastic risk free rates.

Appendix

Solution of the mixed debt problem in some extreme cases

We can verify the obtained solution (31) for the mixed debt case looking at some "limit " or extreme cases. We study the behavior of the solution in three particular cases, considering its particular importance, as well as the fact that its solutions are known. The analysis below of these three cases confirms that our general solution applies in each one of them.

A1. Limit case $B_f = 0$

The whole debt is issued in domestic currency, i.e. $B_f = 0$, hence we have to recover Merton's solution.

Since $B_f = 0$, we get from (31)

$$F(V, X, t) = VI_1 + e^{-(T-t)r_d} B_d I_3 \tag{42}$$

and using again the assumption $B_f = 0$

$$I_{11} = \frac{\ln\left(\frac{V}{B_d}\right) + (r_d + \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}$$

$$I_{33} = \frac{\ln\left(\frac{V}{B_d}\right) + (r_d - \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}$$

since I_{11} and I_{33} (see (35) and (37)) are actually independent of s, we obtain from (32), (34) that

$$I_1 = (1 - N(I_{11}))\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\frac{1}{2}s^2} ds = (1 - N(I_{11}))$$

$$I_3 = N(I_{33})\,\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\frac{1}{2}s^2} ds = N(I_{33})$$

where we have used that $\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\frac{1}{2}s^2} ds = N(\infty) = 1$. And going back to (42) we obtain the well known solution given by Merton, for the whole debt issued in local currency.

A2. Limit case $\mathbf{B}_d = 0$

The whole debt is issued in foreign currency, i.e. $B_d = 0$, hence we have to recover the solution (25) given in section 3.

Our aim is to recover (25). Since $B_d = 0$ (31) yields

$$F(V, X, t) = VI_1 + e^{-(T-t)r_f} B_f X I_2 \tag{43}$$

and using again the condition $B_d = 0$ we obtain from (35) and (36)

$$I_{11} = \frac{\ln\left(\frac{V}{B_f X e^{s\sigma^*\sqrt{T-t} - \frac{\sigma^{*2}}{2}(T-t) + (r_d - r_f)(T-t)}}\right) + (r_d + \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}$$

$$I_{22} = \frac{\ln\left(\frac{V}{B_f X e^{s\sigma^*\sqrt{T-t} + \frac{\sigma^{*2}}{2}(T-t) + (r_d - r_f)(T-t)}}\right) + (r_d - \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}$$

hence by elementary properties of the ln function

$$I_{11} = \frac{\ln\left(\frac{V}{B_f X}\right) + (r_f + \frac{\sigma^2}{2} + \frac{\sigma^{*2}}{2})(T-t)}{\sigma\sqrt{T-t}} - s\frac{\sigma^*}{\sigma}$$

$$I_{22} = \frac{\ln\left(\frac{V}{B_f X}\right) + (r_f - \frac{\sigma^2}{2} - \frac{\sigma^{*2}}{2})(T-t)}{\sigma\sqrt{T-t}} - s\frac{\sigma^*}{\sigma}$$

then

$$N(I_{11}) = N\left(\frac{\ln\left(\frac{V}{B_f X}\right) + (r_f + \frac{\sigma^2}{2} + \frac{\sigma^{*2}}{2})(T-t)}{\sigma\sqrt{T-t}} - s\frac{\sigma^*}{\sigma}\right)$$

$$N(I_{22}) = N\left(\frac{\ln\left(\frac{V}{B_f X}\right) + (r_f - \frac{\sigma^2}{2} - \frac{\sigma^{*2}}{2})(T-t)}{\sigma\sqrt{T-t}} - s\frac{\sigma^*}{\sigma}\right)$$

a glance at (25), (32), (33) and the fact that $\delta^2 = \sigma^2 + \sigma^{*2}$ (since in our case

$\rho = 0$) shows that we are done if we can prove that

$$N(d1) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}s^2} N(I_{11})$$

$$N(d2) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}s^2} N(I_{22}).$$

We are going to prove only the first of this identities since the other follows in the same way.

Let us notice that

$$I_{11} = \frac{\delta}{\sigma} d1 - s\frac{\sigma^*}{\sigma}$$

so defining the auxiliary function $S(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}s^2} N(\frac{\delta}{\sigma}x - s\frac{\sigma^*}{\sigma})ds$ we just need to show that

$$N(x) = S(x). \tag{44}$$

We claim that $N(0) = S(0)$. Indeed, on the one hand $N(0) = \frac{1}{2}$, and on the other, the function $l(s) = N(-s\frac{\sigma^*}{\sigma}) - \frac{1}{2}$ is clearly an odd function. Hence

$$S(0) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}s^2} (l(s) + \frac{1}{2})ds = \frac{1}{2}$$

because of $\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}s^2} l(s)ds = 0$, $\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}s^2} ds = 1$. Then in order to prove (44) it is enough to see that $N'(x) = S'(x)$ (if it is the case then $N(x) - S(x)$ is constant, but $N(0) - S(0) = 0$, and hence $N(x) = S(x)$). Now, from the expression of $S(x)$

$$S'(x) = \frac{\delta}{\sigma} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}s^2} e^{-\frac{1}{2}(\frac{\delta}{\sigma}x - s\frac{\sigma^*}{\sigma})^2} ds$$

and an straightforward computation (using that $\delta^2 = \sigma^2 + \sigma^{*2}$) yields

$$S'(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}$$

but the right hand side is precisely $N'(x)$.

A3. Limit case $\sigma^* = 0$.

The debt is issued in both currencies, although there is not risk on the exchange rate, i.e. $\sigma^* = 0$. In this case the situation obviously can be reduced to the case of an equivalent amount of debt issued in a common currency, just taking into account the difference between both risk free rates.

Under the assumption $\sigma^* = 0$, we get from (35), (36), and (37)

$$I_{11} = \frac{\ln\left(\frac{V}{B_f X e^{(r_d - r_f)(T-t)} + B_d}\right) + (r_d + \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}$$

$$I_{22} = \frac{\ln\left(\frac{V}{B_f X e^{(r_d - r_f)(T-t)} + B_d}\right) + (r_d - \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}$$

$$I_{33} = \frac{\ln\left(\frac{V}{B_f X e^{(r_d - r_f)(T-t)} + B_d}\right) + (r_d - \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}$$

let us notice that $I_{22} = I_{33}$. Hence, calling $I = I_2 = I_3$, we get from (31)

$$F(V, X, t) = V I_1 + (e^{-(T-t)r_f} B_f X + e^{-(T-t)r_d} B_d) I. \tag{45}$$

Where in this case the expressions for I_1 and I takes the form

$$I_1 = N\left(\frac{\ln\left(\frac{V}{B_f X e^{(r_d - r_f)(T-t)} + B_d}\right) + (r_d + \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}\right)$$

$$I = N\left(\frac{\ln\left(\frac{V}{B_f X e^{(r_d-r_f)(T-t)}+B_d}\right)+(r_d-\frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}\right)$$

because of $I_{22} = I_{33}$ are independents of s.

Equation (45) has a plain interpretation. Indeed, if there is not risk on the exchange rate the total amount to be paid at maturity is known with certainty, an it is

$$B_f X e^{c_X(T-t)} + B_d$$

on the other hand an arbitrage argument shows that in this case

$$r_d - r_f = c_X$$

necessarily holds. Then, this amount can be written in domestic currency as

$$B_f X e^{(r_d-r_f)(T-t)} + B_d$$

Since this amount is no longer stochastic the price of the debt could be computed by means of the Merton's approach, and this is what (45) is saying. Indeed, one can easily see that calling $B = B_f X e^{(r_d-r_f)(T-t)} + B_d$, (45) can be written as

$$\begin{aligned} F(V,X,t) &= VN\left(\frac{\ln\left(\frac{V}{B}\right)+(r_d+\frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}\right) \\ &\quad + e^{-(T-t)r_d} BN\left(\frac{\ln\left(\frac{V}{B}\right)+(r_d-\frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}\right) \end{aligned} \qquad (46)$$

i.e. Merton's solution again.

A4. Comparative statics.

Proof of Property 1: Since

$$I_2 = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} N\left(I_{22}\right) e^{-\frac{1}{2}s^2} ds \sim \frac{1}{\sqrt{2\pi}} \int_{-3}^{3} N\left(I_{22}\right) e^{-\frac{1}{2}s^2} ds$$

and

$$I_3 \sim \frac{1}{\sqrt{2\pi}} \int_{-3}^{3} N\left(I_{33}\right) e^{-\frac{1}{2}s^2} ds$$

we get, under the assumption $\sigma^* \sqrt{T-t} \geq 3$, that $\sigma^* \sqrt{T-t} \geq \mid s \mid$ for any $s \in [-3, 3]$, and then is easy to see (by a direct computation) that I_{33} is increasing and I_{22} decreasing w.r.t. σ^*. So, I_3 is increasing in σ^* because $N(\ I_{33})$ is a composition of two increasing functions w.r.t. σ^*. Also, I_2 is decreasing w.r.t. σ^* because $N(I_{22})$ is the composition of an increasing and a dereasing function. Since $\frac{d(f/g)}{dx} > 0$ is valid if f is increasing, g decreasing and both are positive functions, we have just proven the Property 1.

Proof of Property 2

Is analogous to the proof of Property 1, and uses the fact that $I_1$is the composition of an increasing and a decreasing function, and hence decreasing.

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Notes

[1]In further work we intend to analyze our model under more general assumptions on the interest rates, but from now on r_f and r_d are assumed to be constants.

[2]Its price may or not agree with its expected value computed at some previous time t, i.e. $E(B_f X_T + B_d) = B_f X_t e^{c_X(T-t)} + B_d$. The particular case $c_X = 0$ says that X_t is a martingale and hence $E(B_f X_T + B_d) = B_f X_t + B_d$.