# Chapter 3

# On strategy-proofness and single-crossing

## 3.1 Introduction

It is well-known both in modern economic theory and positive political science that voting, in general, can fail to produce well-defined collective outcomes. For instance, the conflict of interests in a society may be such that none of the feasible social alternatives has the support of a majority of voters against any other alternative. Furthermore, it is also known that none of the aggregation methods via voting are free of individual and group manipulation.

To overcome these negative results, it is common in social choice theory to place restrictions on individual preferences. This allows to study the properties of these voting procedures by looking at more homogenous societies. If the social alternatives can be placed over the real line, as for instance when different levels of a public good or different tax rates are the subject of collective choice, one of the most common preference restrictions is *single-crossing*.[1]

This restriction makes sense in many political settings. In few words, a society has single-crossing preferences if, given any two policies, one of them more to the right than the other, the more rightist is an individual (with

[1] The other one is, of course, single-peakedness.

respect to another individual) the more he will prefer the right-wing policy over the left-wing one. For instance, if alternatives are tax rates and individuals are *ordered* according to their income, this restriction means simply that, the richer is an individual the lower will be the tax rate he will prefer.

Technically, this condition not only guarantees the existence of majority voting equilibria, but it also provides a simple characterization of the core of the majority rule. In fact, the core is simply the set of ideal points of the median agent in the ordering of the individuals that makes the preference profile single-crossing. This result is sometimes referred to as the *Representative Voter Theorem* (Rothstein, 1991) or, alternatively, as "the second version" of the *Median Voter Theorem* (Myerson, 1996 and Gans and Smart, 1996).

In any case, the main problem with this result is that, unlike the original Median Voter Theorem over single-peaked preferences, whose non-cooperative foundation was provided by Black (1948), first, and then by Moulin (1980), the Representative Voter Theorem is based on the assumption that individuals honestly reveal their preferences. That is, it is derived assuming *sincere* voting. In effect, even though single-crossing is now largely used in models of collective decision-making, nothing has been said in the literature about the possibility of manipulation (strategic voting) over this preference domain. Moreover, the "single-crossing version" of the Median Voter Theorem is usually applied without caring much about its strategic foundations.

This issue has been considered in the last chapter. It has been shown there that the single-crossing condition guarantees not only majority voting equilibria, but also non-manipulable choice rules. In particular, it showed that this is true for the median choice rule, which is found to be strategy-proof as well as group-strategic-proof. As a by-product, it has also proved that the collective outcome predicted by the Representative Voter Theorem can be implemented in dominant strategies through a simple mechanism. This mechanism is a two-stage voting procedure in which, first, individuals select a representative among themselves, and then the representative voter chooses a policy to be implemented by the planner.

Taken this as a starting point, this chapter characterizes the whole family of strategy-proof social choice functions over the domain of single-crossing preference profiles. The main result shows that this family is completely described by the class of *positional dictator* choice rules; i.e. by all those rules derived from the extended median rule by distributing phantom voters at the extremes of the extended non-negative real line. This class is shown to be strategy-proof as well as group-strategy-proof. Moreover, it is also proved that those rules are non-manipulable not only over the full set of alternatives, but also over every possible policy *agenda*. Interestingly, the chapter shows that, for this kind of individual preferences, the above results cannot be extended to other median voter schemes.

## 3.2 The model, notation and definitions

The basic model of single-crossing preferences assumes that the set of agents $I$ is finite and its cardinality $|I| = n > 2$ is odd. Individuals in $I$ must choose a policy (for example, the level of a given local tax) from a feasible set of social alternatives. They do this by voting.

The set of all possible collective outcomes $X = \{x_1, \ldots, x_l\}$, $|X| > 2$, is assumed to be a finite subset of the extended non-negative real line $\mathbf{R}^*_+ = \Re_+ \cup \{+\infty\}$. The set $X$ is such that $x_j \leq x_k$ for $j \leq k$, where the linear order $\leq$ is the usual order on $\mathbf{R}^*_+$. For a vector $x = (x_1, \ldots, x_n) \in (\mathbf{R}^*_+)^n$, we let $x_{-i} = (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)$ and $(\hat{x}_i, x_{-i}) = (x_1, \ldots, x_{i-1}, \hat{x}_i, x_{i+1}, \ldots, x_n)$, where $\hat{x}_i \in \mathbf{R}^*_+$. In addition, for any group of agents $D \subseteq I$, we denote $(x_D, x_{D^c}) = ((x_i)_{i \in D}, (x_j)_{j \in D^c})$, where $D^c = I \backslash D$.

The set of all feasible alternatives may be either the entire $X$ or just one of its non-empty subsets. The set $\tilde{X}$ represents a generic subset - with the induced order - of $X$. We use $A(X)$ to represent the set of all non-empty subsets of $X$, $A(X) = \{\tilde{X} : \tilde{X} \in 2^X \backslash \emptyset\}$. In words, $X$ is the universal set of outcomes, whereas a particular situation, or *agenda*, involves a $\tilde{X} \in A(X)$.

Let $P(X)$ be the set of all complete, transitive and antisymmetric binary orderings of $X$. We say $P(X)$ is the *universal domain* of individual preferences.[2] Agent $i$'s preferences over the alternatives in $X$ are assumed to

[2]Indifference between alternatives is not allowed. This is a natural assumption when the set of alternatives is finite.

be completely characterized by a single parameter $\theta_i \in \Theta = \{\theta^1, \ldots, \theta^m\}$, where $\Theta \subset \Re$ is a finite and *ordered* subset of the real line, such that $\theta^1 < \theta^2 < \ldots < \theta^m$ and $m \leq |P(X)|$. As usual, we interpret $\theta_i$ as being agent $i$'s *type*.

That is, we assume there exists a function $\Phi : \Theta \to P(X)$ that assigns a unique element $\succ_\theta \in P(X)$ to each $\theta \in \Theta$. We say that $\succ_i$ represents the preferences of an agent $i$ of type $\theta_i$ if,

$$\forall x,\, y \in X,\ x \succ_i y \ \Leftrightarrow\ x\, \Phi(\theta_i)\, y.$$

The maximal set associated with the pair $\langle X, \succ_i\rangle$ is $M(X, \succ_i) = \{x \in X : \forall y \in X \backslash \{x\},\, x \succ_i y\}$. That is, $M(X, \succ_i)$ yields the alternative that is top-ranked in $X$ for $i$ with respect to her preferences $\succ_i$. Notice that since preferences are strict, maximal sets are indeed singletons.

A preference profile associated to a profile of types $\theta = (\theta_1, \ldots, \theta_n) \in \Theta^n$ is an $n$-tuple $(\succ_1, \ldots, \succ_n) = (\Phi(\theta_1), \ldots, \Phi(\theta_n))$ in $P(X)^n$. This means that the profile of individual preferences depends on the *state* $\theta \in \Theta^n$: in the state $\theta$, agent $i$ has preferences $\Phi(\theta_i)$ over the set $X$. This formulation allows for any degree of correlation across the agents' preferences. We assume each agent observes $\theta$, so that there exists complete information among the agents about their preferences over $X$. Extending our earlier conventions to preference profiles, we have that $\succ_{-i} = (\succ_1, \ldots, \succ_{i-1}, \succ_{i+1}, \ldots, \succ_n)$. Similarly, the profile obtained by changing agent $i$'s preferences for $\hat{\succ}_i$ is $(\hat{\succ}_i, \succ_{-i}) = (\succ_1, \ldots, \succ_{i-1}, \hat{\succ}_i, \succ_{i+1}, \ldots, \succ_n)$. Finally, for any group of agents $D \subseteq I$, $(\succ_D, \succ_{D^c}) = ((\succ_i)_{i \in D}, (\succ_j)_{j \in D^c})$.

Now, we restrict the set of admissible preference profiles by imposing a condition on preferences that involves the entire profile:

**Definition 1** *A preference profile $(\succ_1, \ldots, \succ_n)$ derived from $\Phi : \Theta \to P(X)$ is single-crossing on $X$ if, for all $x$, $y \in X$ and all $i$, $j \in I$ such that either $y > x$ and $\theta_j > \theta_i$ or $y < x$ and $\theta_j < \theta_i$,*

$$y\, \Phi(\theta_i)\, x \quad \Rightarrow \quad y\, \Phi(\theta_j)\, x.$$

We denote $SC(X)$ the set of all single-crossing preference profiles on $X$.[3] The recent interest on this restricted domain of preferences is due to the fact that, like single-peakedness,[4] single-crossing has been shown to be sufficient to guarantee the existence of majority voting equilibria. However, apart from this fact, it should be clear that both domain conditions are independent, in the sense that neither property is logically implied by the other. Tables 3.1, 3.2 and 3.3 below show three situations (for the simplest possible case of three individuals and three alternatives) in which preferences are, respectively, (1) single-crossing, but not single-peaked; (2) single-peaked, but not single-crossing; and (3) both single-crossing and single-peaked:

Table 3.1: Single-crossing

| $\Phi(\theta^1)$ | $\Phi(\theta^2)$ | $\Phi(\theta^3)$ |
|---|---|---|
| $x_1$ | $x_1$ | $x_3$ |
| $x_2$ | $x_3$ | $x_2$ |
| $x_3$ | $x_2$ | $x_1$ |

Table 3.2: Single-peakedness

| $\succ_1$ | $\succ_2$ | $\succ_3$ |
|---|---|---|
| $x_1$ | $x_4$ | $x_2$ |
| $x_2$ | $x_2$ | $x_1$ |
| $x_4$ | $x_1$ | $x_3$ |
| $x_3$ | $x_3$ | $x_4$ |

[3]Other expressions used in the literature to denominate similar preference restrictions are *hierarchical adherence*, *order-restriction* and *unidimensional alignment*. For more on them, see Roberts (1977), Rothstein (1990, 1991), Gans and Smart (1996), Austen-Smith and Banks (1999) and List (2001), and the references quoted there.

[4]Formally, a preference profile $(\succ_1, \ldots, \succ_n)$ is single-peaked on $X$ with respect to the linear order $\leq$ if for all $i \in I$, there exists $\tau_i \in X$, called the *peak* of $i$ associated to the preference relation $\succ_i$, such that (1) $\tau_i \succ_i x$, for all $x \in X \backslash \{\tau_i\}$; (2) $y < x \leq \tau_i$ implies $x \succ_i y$, and (3) $\tau_i \leq x < y$ implies $x \succ_i y$.

Table 3.3: Single-crossing and single-peakedness

| $\Phi(\theta^1)$ | $\Phi(\theta^2)$ | $\Phi(\theta^3)$ |
|---|---|---|
| $x_1$ | $x_2$ | $x_3$ |
| $x_2$ | $x_1$ | $x_2$ |
| $x_3$ | $x_3$ | $x_1$ |

In the political arena, single-crossing makes sense in many applications. For instance, suppose individual types are interpreted as being different ideological characters, arranged in the left-right scale, and the alternatives as public policies to be chosen by the society. Then, preferences are single-crossing if, for any two policies, one of them more to the right than the other, the more rightist is a type, the more will he prefer the right-wing policy over the left-wing one.

Given a preference $\succ_i$ in the profile $\succ \in SC(X)$, we define agent $i$'s *induced* preferences over the agenda $\tilde{X} \in A(X)$, $\tilde{\succ}_i$, as follows:

$$\forall x,\, y \in \tilde{X},\; x \,\tilde{\succ}_i\, y \;\Leftrightarrow\; x \succ_i y.$$

Notice that the property of being single-crossing is preserved in the induced preferences. That is, if $\succ \in SC(X)$ then $\tilde{\succ} \in SC(\tilde{X})$, for all $\tilde{X} \in A(X)$.

These preferences can be aggregated. The input for this aggregation process is the set of *declarations* of the individuals. These declarations are intended to provide information about their true types, although their sincerity cannot be ensured.

The aggregation process is represented by a social choice function. For any $\tilde{X} \in A(X)$, a *social choice function* $f$ on $SC(\tilde{X})$ is a single-value mapping $f : SC(\tilde{X}) \rightarrow \tilde{X}$ that associates to each preference profile $\tilde{\succ} = (\tilde{\succ}_1, \ldots, \tilde{\succ}_n) \in SC(\tilde{X})$ a unique outcome $f(\tilde{\succ}) \in \tilde{X}$.

We will be interested in social choice functions that satisfy the following properties. The main one is that agents, acting individually or in groups, never have the incentives to misrepresent their preferences. To capture this idea, we define the following two concepts:

**Definition 2** *A social choice function $f$ on $SC(\tilde{X})$ is strategy-proof if for all $\tilde{\succ} \in SC(\tilde{X})$, and for any agent $i \in I$, with type $\theta_i$, any misrepresentation $\hat{\succ}_i = \tilde{\Phi}(\hat{\theta}_i)$, $\hat{\theta}_i \neq \theta_i$, is such that either $f(\tilde{\succ}) \,\tilde{\succ}_i\, f(\hat{\succ}_i, \tilde{\succ}_{-i})$ or $f(\tilde{\succ}) = f(\hat{\succ}_i, \tilde{\succ}_{-i})$, where $(\hat{\succ}_i, \tilde{\succ}_{-i}) \in SC(\tilde{X})$.*[5]

If a social choice function $f$ is not strategy-proof, then there exist $i \in I$ and $\hat{\succ}_i$ such that for some $\tilde{\succ}_{-i}$, $(\hat{\succ}_i, \tilde{\succ}_{-i}) \in SC(\tilde{X})$, and $i$'s true preferences, $\tilde{\succ}_i$, $f(\hat{\succ}_i, \tilde{\succ}_{-i}) \,\tilde{\succ}_i\, f(\tilde{\succ}_i, \tilde{\succ}_{-i})$. Then, we say $f$ is *manipulable* at $(\tilde{\succ}_i, \tilde{\succ}_{-i})$, by $i$, via $\hat{\succ}_i$. In the same way:

**Definition 3** *A social choice function $f$ on $SC(\tilde{X})$ is group-strategy-proof if for all $\tilde{\succ} \in SC(\tilde{X})$, and for every coalition $D \subseteq I$, with types $\theta_D = (\theta_i)_{i \in D}$, there does not exist a joint misrepresentation $\hat{\succ}_D = (\tilde{\Phi}(\hat{\theta}_i))_{i \in D}$, $\hat{\theta}_D \neq \theta_D$, such that, for all $i \in D$, $f(\hat{\succ}_D, \tilde{\succ}_{D^c}) \,\tilde{\succ}_i\, f(\tilde{\succ})$, where $(\hat{\succ}_D, \tilde{\succ}_{D^c}) \in SC(\tilde{X})$.*

Another crucial property we may seek in a social choice function is *Pareto efficiency.* This condition is well-known and requires no further comment here:

**Definition 4** *A social choice function $f$ on $SC(\tilde{X})$ is Pareto efficient if and only if, for all $\tilde{\succ} \in SC(\tilde{X})$,*

$$f(\tilde{\succ}) \in \{x \in \tilde{X} : \nexists\, y \in \tilde{X} \text{ such that } y \,\tilde{\succ}_i\, x \;\forall i \in I\}.$$

One last property a social choice function may satisfy is *tops-onliness.* We say that $f$ is *tops-only* if for any profile of preferences $\tilde{\succ} \in SC(\tilde{X})$ the social outcome $f(\tilde{\succ})$ is determined only by the individuals' most-preferred alternatives in $\tilde{\succ}$. Formally, for any individual ordering $\tilde{\succ}_i$ in $\tilde{\succ} \in SC(\tilde{X})$, let $\tau(\tilde{\succ}_i) = M(\tilde{X}, \tilde{\succ}_i)$:

**Definition 5** *A social choice function $f$ on $SC(\tilde{X})$ is tops-only if for any two preference profiles $\tilde{\succ}$ and $\hat{\succ}$ in $SC(\tilde{X})$, such that for any $i \in I$, $\tau(\tilde{\succ}_i) = \tau(\hat{\succ}_i)$, $f(\tilde{\succ}) = f(\hat{\succ})$.*

[5] With $\tilde{\Phi}(\hat{\theta}_i)$ we represent the restriction of $\Phi(\hat{\theta}_i)$ over $\tilde{X}$.

Of course, the tops-only property dramatically constraints the scope for manipulation: no agent can expect to be able to affect the social outcome without modifying the peak of his reported preference ordering. However, as we will show, this condition is related to the strategy-proofness condition itself. In effect, when preferences are single-crossing, it turns out that every strategy-proof social choice rule whose range is greater than two must be tops-only (see Corollary 3 below).

We now define the *extended median rule*. This social choice function is a particular member of the class of *anonymous* and tops-only choice rules,[6] which provides a natural extension of the basic idea of the *median choice rule*.

For any odd positive integer $k$, let $m^k : (\mathbf{R}^*_+)^k \to \mathbf{R}^*_+$ be the *k-median function*, defined in the following way: for all $x \in (\mathbf{R}^*_+)^k$, $m^k(x)$ is the *k-median* of $x = (x_1, \ldots, x_k)$ if and only if $|\{x_i \in \mathbf{R}^*_+ : x_i \leq m^k(x)\}| \geq \frac{(k+1)}{2}$ and $|\{x_j \in \mathbf{R}^*_+ : m^k(x) \leq x_j\}| \geq \frac{(k+1)}{2}$. Because $k$ is odd, this function is always well-defined. Now, we define the *extended median rule* in the following way:

**Definition 6** *A social choice function $f^e$ on $SC(\tilde{X})$ is called the extended median rule if there exist $n+1$ real numbers $\alpha_1, \ldots, \alpha_{n+1} \in \mathbf{R}^*_+$, called the* phantom voters*, such that, for all $\tilde{\succ} \in SC(\tilde{X})$,*

$$f^e(\tilde{\succ}) = m^{2n+1}(\tau(\tilde{\succ}_1), \ldots, \tau(\tilde{\succ}_n),\, \alpha_1, \ldots, \alpha_{n+1}).$$

A particular case of this rule is the following. Let $\alpha_1 = \ldots = \alpha_{\frac{n+1}{2}} = 0$ and $\alpha_{\frac{n+1}{2}+1} = \ldots = \alpha_{n+1} = +\infty$. Then,

$$f^e(\tilde{\succ}) = m^{2n+1}(\tau(\tilde{\succ}_1), \ldots, \tau(\tilde{\succ}_n),\ \underbrace{0, \ldots, 0}_{\frac{(n+1)}{2}\text{ times}},\underbrace{+\infty, \ldots, +\infty}_{\frac{(n+1)}{2}\text{ times}}),$$

is the well-known *median choice rule*, $f^m$, that can be re-written as

$$f^m(\tilde{\succ}) = m^n(\tau(\tilde{\succ}_1), \ldots, \tau(\tilde{\succ}_n)).$$

[6]A social choice function $f$ on $SC(\tilde{X})$ is *anonymous* if for any $\tilde{\succ}$ and $\hat{\succ}$ in $SC(\tilde{X})$, such that $\hat{\succ}$ is a permutation of $\tilde{\succ}$, $f(\tilde{\succ}) = f(\hat{\succ})$.

Proceeding in the same way, other *supermajority* rules can also be derived from $f^e$, by restricting the parameters $\alpha_1, \ldots, \alpha_{n+1}$ to take some particular values in $\mathbf{R}^*_+$. Notice that, if $\alpha_1 =, \ldots, = \alpha_{n+1} = \alpha$, $f^e$ is completely insensitive to the preferences reported by the individuals, since $\forall \tilde{\succ} \in SC(\tilde{X})$

$$f^e(\tilde{\succ}) = m^{2n+1}(\tau(\tilde{\succ}_1), \ldots, \tau(\tilde{\succ}_n), \underbrace{\alpha, \ldots, \alpha}_{(n+1) \text{ times}}) = \alpha.$$

We might want to exclude such undesirable voting rules and, in particular, require Pareto efficiency. In order to allow the extended median rule $f^e$ to satisfy Pareto efficiency, we eliminate the possibility of inefficiency by setting $\alpha_n = 0$ and $\alpha_{n+1} = +\infty$. Therefore, we obtain the following restriction of $f^e$:

$$f^{e^*}(\tilde{\succ}) = m^{2n-1}(\tau(\tilde{\succ}_1), \ldots, \tau(\tilde{\succ}_n), \alpha_1, \ldots, \alpha_{n-1}),$$

which is the *efficient extended median rule* with $n-1$ parameters.

In the following section, we will study how well the extended median rule performs, according to the manipulation criteria given above, on the domain of single-crossing preference profiles.

## 3.3 Main results

Suppose phantom voters are restricted to having peaks at either zero or infinity. That is, assume that, for any $i = 1, \ldots, n-1$, $\alpha_i \in \{0, +\infty\}$, such that each phantom voter is either a *leftist* or a *rightist*. For this particular case in which all ficticious voters take only the extreme values on $\mathbf{R}^*_+$, the Condorcet winners obtained are the well-known class of *positional dictators*.[7]

These rules select the $j$th ranked peak among the tops of the reported preference orderings, for some $j = 1, \ldots, n$. For example, if $j = 1$, we have the *leftist rule*, which chooses the smallest reported peak of a real voter. Of course, the median rule is also a particular case. It turns out that all these rules are group-strategy-proof over $SC(\tilde{X})$, for any $\tilde{X} \in A(X)$:[8]

[7]See Moulin (1988), pp. 302.

[8]To put the phantoms at some point that coincides with the peak of some actual type of the voters, in addition to at zero or infinity, yields the same results. However, we ruled out

**Proposition 1** *Let* $\alpha_1, \ldots, \alpha_{n-1}$ *be such that* $\alpha_1, \ldots, \alpha_{n-1} \in \{0, +\infty\}$. *Then, the extended median rule* $f^{e^*}$ *is group-strategy-proof over* $SC(\tilde{X})$*, for any* $\tilde{X} \in A(X)$.[9]

PROOF Consider a profile $\tilde{\succ} = (\tilde{\succ}_1, \ldots, \tilde{\succ}_n)$ in $SC(\tilde{X})$, with associated types $(\theta_1, \ldots, \theta_n)$. Suppose there exists a coalition $D \subseteq I$ and a list of alternative types for members of $D$, $(\hat{\theta}_i)_{i \in D}$, $(\hat{\theta}_i)_{i \in D} \neq (\theta_i)_{i \in D}$, such that the joint declaration generated by $(\hat{\theta}_i)$, $\hat{\succ}_D = (\tilde{\Phi}(\hat{\theta}_i))_{i \in D}$, produces a preferred social outcome for every member of the coalition. That is, for all $i \in D$,

$$f^{e^*}(\hat{\succ}_D, \tilde{\succ}_{D^c}) \,\tilde{\succ}_i\, f^{e^*}(\tilde{\succ}_D, \tilde{\succ}_{D^c}),$$

where $(\hat{\succ}_D, \tilde{\succ}_{D^c}) \in SC(\tilde{X})$. For simplicity, call $f^{e^*}(\tilde{\succ}) = \tau$ and $f^{e^*}(\hat{\succ}_D, \tilde{\succ}_{D^c}) = \hat{\tau}$. Notice that, by the assumed distribution of the phantom voters, $\tau$ and $\hat{\tau}$ must coincide with the tops reported by some "real" voters. Denote these agents $j$ and $j'$ and their types $\theta_j$ and $\theta_{j'}$, respectively. Since $\tau \neq \hat{\tau}$, assume that $\tau < \hat{\tau}$. Then, for all $i \in D$, $\tau(\tilde{\succ}_i) > \tau$. Suppose not. That is, assume $\tau(\tilde{\succ}_i) \leq \tau$ for some agent $i$ in $D$. If $\tau(\tilde{\succ}_i) = \tau$, then $\tau \,\tilde{\succ}_i\, \hat{\tau}$, which contradicts our hypothesis. Consider, instead, that $\tau(\tilde{\succ}_i) < \tau$. Since $\hat{\tau} \,\tilde{\succ}_i\, \tau$, by single-crossing we have that for all $\theta > \theta_i$, $\hat{\tau}\, \tilde{\Phi}(\theta)\, \tau$. Then, $\theta_j$ has to verify that $\theta_j < \theta_i$ and, by single-crossing, $\tau\, \tilde{\Phi}(\theta_j)\, \tau(\tilde{\succ}_i)$ implies $\tau\, \tilde{\Phi}(\theta_i)\, \tau(\tilde{\succ}_i)$. Contradiction. Then, $\tau(\tilde{\succ}_i) > \tau$, for all $i \in D$. The rest of the proof is as follows. By definition,

$$f^{e^*}(\tilde{\succ}_D, \tilde{\succ}_{D^c}) = m^{2n-1}(\tau(\tilde{\succ}_1), \ldots, \tau(\tilde{\succ}_n), \alpha_1, \ldots, \alpha_{n-1}) = \tau,$$

while

$$f^{e^*}(\hat{\succ}_D, \tilde{\succ}_{D^c}) = m^{2n-1}(\{\tau(\hat{\succ}_i)\}_{i \in D}, \{\tau(\tilde{\succ}_j)\}_{j \in D^c}, \alpha_1, \ldots, \alpha_{n-1}) = \hat{\tau}.$$

Two cases are possible:

this possibility for two reasons. First, because then the phantoms and, therefore, the social choice function would depend on the particular profile of preferences considered. Second, and more important, because otherwise to define the choice rule the planner would require information about the actual location of the true tops. But this is precisely one of the problems that he tries to solve by means of the voting process.

[9] A similar result holds for $f^e$. That is, efficiency may be dropped without altering the result of Proposition 1.

1. For each $i \in D$, $\tau(\hat{\succ}_i) > \tau$. Then $\hat{\tau} = \tau$. Contradiction.

2. For some $i \in D$, $\tau(\hat{\succ}_i) \leq \tau$. Then, if we rename $(\{\tau(\hat{\succ}_i)\}_{i \in D}, \{\tau(\tilde{\succ}_j)\}_{j \in D^c}, \alpha_1, \ldots, \alpha_{n-1})$ as $(y_1, \ldots, y_{2n-1})$, we have that

$$|\{j \in \{1, \ldots, (2n-1)\} : y_j \leq \tau\}| \geq n.$$

But this implies that $m^{2n-1}(y_1, \ldots y_{2n-1}) \leq \tau$. That is, $f(\hat{\succ}_D, \tilde{\succ}_{D^c}) \leq f(\tilde{\succ}_D, \tilde{\succ}_{D^c})$, which contradicts our initial hypothesis. 2

Thus, falling short of Moulin's (1980) results, Proposition 1 shows that efficient and anonymous generalized median voter schemes are group-strategy-proof (and consequently, strategy-proof) over single-crossing preference profiles, provided that the phantom voters are fixed at the extremes of $\mathbf{R}^*_+$, (i.e., at 0 or $+\infty$).

Interestingly, strategy-proofness cannot be guaranteed in the case of other extended median rules, which allow the socially selected alternative to be the top of a fictitious voter. This conclusion applies also, of course, to the case in which the social choice rule violates the Pareto condition. The following example illustrates this point:[10]

**Example 1** *Consider two possible preference profiles* $(\tilde{\succ}_1, \ldots, \tilde{\succ}_n)$ *and* $(\tilde{\succ}_{-n}, \hat{\succ}_n)$ *in* $SC(\tilde{X})$*, and the corresponding collective outcomes* $f^e(\tilde{\succ}) = \tau$ *and* $f^e(\tilde{\succ}_{-n}, \hat{\succ}_n) = \hat{\tau}$*, where* $\hat{\tau} < \tau$*. Suppose that individual preferences are such that, for each individual* $i \in I$*,* $\hat{\tau} \tilde{\succ}_i \tau$*. For instance, set* $\tilde{\succ}_i = \tilde{\succ}_1$ *for all* $i \in I$*,* $i \neq n$*, with* $\tau(\tilde{\succ}_1) = \hat{\tau}$*, and assume that the true preferences of agent* $n$ *over* $\tilde{X}$*,* $\tilde{\succ}_n$*, are such that* $\tau(\tilde{\succ}_n) > \tau$ *(see Figure 3.1 below). Notice that* $\tau$ *does not coincide with the most-preferred alternative of any of the agents. Then, set* $\alpha_1 = \ldots = \alpha_{n-1} = +\infty$*,* $\alpha_n = \tau$ *and* $\alpha_{n+1} = 0$*. It is clear that:*

$$f^e(\tilde{\succ}_1, \ldots, \tilde{\succ}_n) = m^{2n+1}(\underbrace{\hat{\tau}, \ldots, \hat{\tau}}_{n-1 \text{ times}}, \tau(\tilde{\succ}_n), \underbrace{+\infty, \ldots, +\infty}_{n-1 \text{ times}}, \tau, 0) = \tau.$$

[10] Of course, this does not occur if phantoms are not restricted to be at zero or plus infinity, but some or all of them are also allowed to be at the tops of some real voters. However, we ruled out this possibility by considering the phantoms fixed parameters, that do not depend on the preference profile.

*Furthermore, it is also evident that the whole coalition $I$ can improve by declaring $(\tilde{\succ}_{-n}, \hat{\succ}_n) \in SC(\tilde{X})$, with $\tau(\hat{\succ}_n) = \hat{\tau}$, since*

$$m^{2n+1}(\underbrace{\hat{\tau}, \dots, \hat{\tau}}_{n \text{ times}}, \underbrace{+\infty, \dots, +\infty}_{n-1 \text{ times}}, \tau, 0) = \hat{\tau},$$

*which is preferred by every coalition member to $\tau$. That such declaration exists is easy to check. Just consider the case in which agent $n$ mimics any of the other agents, so that $\hat{\succ}_n = \tilde{\succ}_1$. However, the joint declaration $(\tilde{\succ}_{-n}, \hat{\succ}_n)$ implies agent $n$ is not revealing honestly his preferences.*

[Insert Figure 3.1 about here]

The reason why strategy-proofness is not preserved in general for the extended median rule, for any possible distribution of the phantoms, is simple. For such arbitrary distributions, the socially selected outcome is not guaranteed to be the most-preferred alternative of a real type. But, without this condition, single-crossing is unable to rule out individual or group manipulations. This is an important difference with single-peakedness, where strategy-proofness is valid both at the individual and the group level, without any restriction on the values of the phantom voters.

Furthermore, it implies that the family of strategy-proof social choice functions on the domain of single-crossing preference profiles is strictly smaller than the same class on single-peakedness. The rest of the paper is dedicated to prove this result.

**Theorem 1** *If $f : SC(\tilde{X}) \to \tilde{X}$ is a tops-only, efficient and strategy-proof social choice function, there exist $\alpha_1, \dots, \alpha_{n-1} \in \{0, +\infty\}$ such that for every profile $\tilde{\succ} \in SC(\tilde{X})$:*

$$f(\tilde{\succ}) = m^{2n-1}(\tau(\tilde{\succ}_1), \dots, \tau(\tilde{\succ}_n), \alpha_1, \dots, \alpha_{n-1}). \qquad (*)$$

Proof Suppose by contradiction that for every combination $\alpha_1, \dots, \alpha_{n-1} \in \{0, +\infty\}$ there exists a profile $\tilde{\succ} \in SC(\tilde{X})$ such that $f(\tilde{\succ}) \neq m^{2n-1}(\tau(\tilde{\succ}_1), \dots, \tau(\tilde{\succ}_n), \alpha_1, \dots, \alpha_{n-1})$. That is equivalent to claim that, if we denote by $i^*$ the $i$-th position in the order of declarations, for every $i^* = 1, \dots, n$, there exists a profile $\tilde{\succ} \in SC(\tilde{X})$ such that

$$f(\tilde{\succ}) \neq \tau(\tilde{\succ}_{i^*}). \quad (**)$$

where, as said, agent $i^*$ is the individual whose peak takes up the $i$-th place (according to the linear order $\leq$) in the distribution of tops $\tau(\tilde{\succ}_{1^*}), \ldots, \tau(\tilde{\succ}_{i-1^*}), \tau(\tilde{\succ}_{i^*}), \tau(\tilde{\succ}_{i+1^*}), \ldots, \tau(\tilde{\succ}_{n^*})$ generated by the profile $\tilde{\succ} \in SC(\tilde{X})$.[11] Otherwise, if there were a position, say the $i$-th, such that for every $\tilde{\succ} \in SC(\tilde{X})$, $f(\tilde{\succ}) = \tau(\tilde{\succ}_{i^*})$ we would get a contradiction, since

$$\tau(\tilde{\succ}_{i^*}) = m^{2n-1}(\tau(\tilde{\succ}_{1^*}), \ldots, \tau(\tilde{\succ}_{i^*}), \ldots, \tau(\tilde{\succ}_{n^*}), \underbrace{0, \ldots, 0}_{n-j \text{ times}}, \underbrace{+\infty, \ldots, +\infty}_{j-1 \text{ times}}).$$

Thus, consider the $i$-th position and a profile $\tilde{\succ} \in SC(\tilde{X})$ verifying $(**)$. Then, if $f(\tilde{\succ}) = x$, $x \neq \tau(\tilde{\succ}_{i^*})$, where $\tau(\tilde{\succ}_{i^*})$ is as before the peak ranked in the $i$-th place. For an agent $k$, consider two alternative preferences, $\hat{\succ}_k$ and $\bar{\succ}_k$, such that they verify *simultaneously* the following properties:

*Property 1:* Both $(\hat{\succ}_k, \tilde{\succ}_{-k})$ and $(\bar{\succ}_k, \tilde{\succ}_{-k})$ are in $SC(\tilde{X})$.
*Property 2:* $f(\hat{\succ}_k, \tilde{\succ}_{-k}) = y \neq x$.
*Property 3:* $\tau(\bar{\succ}_k) = \tau(\tilde{\succ}_k)$.
*Property 4:* $f(\hat{\succ}_k, \tilde{\succ}_{-k}) \, \bar{\succ}_k \, f(\tilde{\succ})$.

The existence of a $k$ and the corresponding binary orderings $\hat{\succ}_k$ and $\bar{\succ}_k$ is ensured by Lemmas 1-2 in the Appendix. Therefore we have a pair of preferences $\hat{\succ}_k$ and $\bar{\succ}_k$ such that $(\hat{\succ}_k, \tilde{\succ}_{-k})$, $(\bar{\succ}_k, \tilde{\succ}_{-k}) \in SC(\tilde{X})$, while $f(\hat{\succ}_k, \tilde{\succ}_{-k}) \neq f(\tilde{\succ})$ and $\tau(\bar{\succ}_k) = \tau(\tilde{\succ}_k)$. Since $f$ is tops-only, $f(\tilde{\succ}) = f(\bar{\succ}_k, \tilde{\succ}_{-k})$. Then, $f(\hat{\succ}_k, \tilde{\succ}_{-k}) \, \bar{\succ}_k \, f(\tilde{\succ})$ implies that $f(\hat{\succ}_k, \tilde{\succ}_{-k}) \, \bar{\succ}_k \, f(\bar{\succ}_k, \tilde{\succ}_{-k})$. But this contradicts the assumption that $f$ is strategy-proof. Summarizing, we derived a contradiction from assuming $(**)$. This means there exists a combination $\alpha_1, \ldots, \alpha_{n-1} \in \{0, +\infty\}$ such that $(*)$ holds. 2

[11]Notice that the $i$-th position in the above distribution may not be occupied by the agent indexed by $i$. Single-crossing admits situations where this is the case. Hence, it is important to distinguish between the index of the agent and the position its peak has in the distribution of tops. For notational simplicity we will omit the distinction wherever it is not relevant.

Let $A_f$ be the range of the social choice function $f : SC(\tilde{X}) \to \tilde{X}$. Then,

**Theorem 2** *A social choice function $f : SC(\tilde{X}) \to \tilde{X}$, with range $|A_f| > 2$, is tops-only and strategy-proof if and only if there exists $\alpha_1, \ldots, \alpha_{n-1} \in \{0, +\infty\}$ such that, for all $\tilde{\succ} \in SC(\tilde{X})$,*[12]

$$f(\tilde{\succ}) = m^{2n-1}(\tau(\tilde{\succ}_1), \ldots, \tau(\tilde{\succ}_n), \alpha_1, \ldots, \alpha_{n-1}).$$

Proof
(⇐): Immediate from Proposition 1.
(⇒): As the proof of Theorem 1, but using Lemmas 1 and 3 of the Appendix.
2

Given a profile $(\succ_1, \ldots, \succ_n) \in SC(X)$ and an arbitrary subset $\tilde{X} \in A(X)$, denote $T(\tilde{X}, \tilde{\succ}) = \{x \in \tilde{X} : \exists i \in I$ such that $\tau(\tilde{\succ}_i) = x\}$ the set of all individual peaks in $\tilde{X}$ generated by the induced profile $(\tilde{\succ}_1, \ldots, \tilde{\succ}_n)$:

**Corollary 1** *If $f : SC(\tilde{X}) \to \tilde{X}$ is a tops-only and strategy-proof social choice function and $|A_f| > 2$, then $f(\tilde{\succ}) \in T(\tilde{X}, \tilde{\succ})$.*

Proof Immediate from Theorem 2. 2

**Corollary 2** *If $f : SC(\tilde{X}) \to \tilde{X}$ is a tops-only and strategy-proof social choice function and $|A_f| > 2$, then $f$ is efficient.*

Proof Trivial. By contradiction, suppose $f : SC(\tilde{X}) \to \tilde{X}$ is a tops-only and strategy-proof social choice function, but assume it is not efficient. Then, there exists a profile $\tilde{\succ} \in SC(\tilde{X})$ and a pair of alternatives $x, y \in \tilde{X}$, $x \neq y$, such that $f(\tilde{\succ}) = x$ while $y \,\tilde{\succ}_i\, x$ for every $i \in I$. Therefore, $f(\tilde{\succ}) \notin T(\tilde{X}, \tilde{\succ})$. But this contradicts Corollary 1. 2

**Proposition 2** *If $f : SC(\tilde{X}) \to \tilde{X}$ is a strategy-proof social choice function, then $f$ is unanimous on its range. That is, for every $\tilde{\succ} \in SC(\tilde{X})$ and any $x \in A_f$, if $\tau(\tilde{\succ}_i) = x \;\forall i \in I$, then $f(\tilde{\succ}) = x$.*

[12]The rules $f$ such that $|A_f| = 1$ are trivially tops-only and strategy-proof, but their (unique) outcomes coincide with those of the extended median rule $f^e$, with its $n+1$ phantoms ranging freely over $\mathbf{R}^*_+$. That is, their outcomes may not fall in the restricted class of the tops of the individual preferences.

PROOF Immediate from the argument $\mathbf{P_2'}$ in Lemma 3 in the Appendix. 2

**Theorem 3** *A social choice function* $f : SC(\tilde{X}) \to \tilde{X}$*, with range* $|A_f| > 2$*, is strategy-proof if and only if there exist* $\alpha_1, \ldots, \alpha_{n-1} \in \{0, +\infty\}$ *such that, for all* $\tilde{\succ} \in SC(\tilde{X})$*,*

$$f(\tilde{\succ}) = m^{2n-1}(\tau(\tilde{\succ}_1), \ldots, \tau(\tilde{\succ}_n), \alpha_1, \ldots, \alpha_{n-1}).$$

PROOF
($\Leftarrow$): Immediate from Proposition 1.
($\Rightarrow$): As the proof of Theorem 1, but using Lemmas 1 and 4 in the Appendix.
2

**Corollary 3** *If* $f : SC(\tilde{X}) \to \tilde{X}$ *is a strategy-proof social choice function and* $|A_f| > 2$*, then* $f$ *is tops-only.*

PROOF Trivial. By contradiction, suppose there exists $\hat{\succ}$ and $\tilde{\succ}$ in $SC(\tilde{X})$, such that $\tau(\hat{\succ}_i) = \tau(\tilde{\succ}_i)$ for all $i \in I$, while $f(\hat{\succ}) \neq f(\tilde{\succ})$. By Theorem 3, there exists $\alpha_1, \ldots, \alpha_{n-1} \in \{0, +\infty\}$ such that,

$$f(\tilde{\succ}) = m^{2n-1}(\tau(\tilde{\succ}_1), \ldots, \tau(\tilde{\succ}_n), \alpha_1, \ldots, \alpha_{n-1}).$$

while

$$f(\hat{\succ}) = m^{2n-1}(\tau(\hat{\succ}_1), \ldots, \tau(\hat{\succ}_n), \alpha_1, \ldots, \alpha_{n-1}).$$

Since $\tau(\hat{\succ}_i) = \tau(\tilde{\succ}_i)$ for each $i \in I$, we have that $f(\hat{\succ}) = f(\tilde{\succ})$. Contradiction. 2

## 3.4 Appendix

**Lemma 1** *Given* $(\tilde{\Phi}(\theta_1), \ldots, \tilde{\Phi}(\theta_n)) \in SC(\tilde{X})$, *there exists* $k \in I$ *and* $\hat{\theta}_k$, $\bar{\theta}_k \in \Theta \backslash \{\theta_k\}$, $\hat{\theta}_k \neq \bar{\theta}_k$, *such that* $(\tilde{\Phi}(\hat{\theta}_k), \{\tilde{\Phi}(\theta_i)\}_{i \neq k}) \in SC(\tilde{X})$ *and* $(\tilde{\Phi}(\bar{\theta}_k), \{\tilde{\Phi}(\theta_i)\}_{i \neq k}) \in SC(\tilde{X})$, *for any* $\tilde{X} \in A(X)$.

Proof Consider the set $\tilde{X} = \{x_1, \ldots, x_s\}$, $s > 2$, and the profile $(\tilde{\Phi}(\theta_1), \ldots, \tilde{\Phi}(\theta_n)) = (\tilde{\succ}_1, \ldots, \tilde{\succ}_n) \in SC(\tilde{X})$. By contradiction, suppose that $\forall k \in I$, $\hat{\theta}_k, \bar{\theta}_k \in \Theta \backslash \{\theta_k\}$, $\hat{\theta}_k \neq \bar{\theta}_k$, either

$$(\tilde{\Phi}(\hat{\theta}_k), \{\tilde{\Phi}(\theta_i)\}_{i \neq k}) \notin SC(\tilde{X}) \text{ or } (\tilde{\Phi}(\bar{\theta}_k), \{\tilde{\Phi}(\theta_i)\}_{i \neq k}) \notin SC(\tilde{X}). \quad (\star)$$

Let $\Theta^I(\tilde{\succ}) = \{\theta \in \Theta : \exists i \in I \text{ such that } \tilde{\succ}_i = \tilde{\Phi}(\theta)\}$ be the set of *actual* types. For a type $\theta_k \in \Theta^I$, let $L(\theta_k) = \{\theta_i \in \Theta^I : \theta_i < \theta_k\}$ and $H(\theta_k) = \{\theta_i \in \Theta^I : \theta_i > \theta_k\}$. It is straightforward to see that, if $|\Theta^I(\tilde{\succ})| > 2$ it is always possible to find a $\theta_k \in \Theta^I(\tilde{\succ})$ such that $H(\theta_k) \neq \emptyset$ and $L(\theta_k) \neq \emptyset$. Then, define $\theta^{\max} = \min_{(\theta)} H(\theta_k)$ and $\theta^{\min} = \max_{(\theta)} L(\theta_k)$. Clearly, $\tilde{\Phi}(\theta_k)$ and $\tilde{\Phi}(\theta^{\min})$ must differ, as well as $\tilde{\Phi}(\theta_k)$ and $\tilde{\Phi}(\theta^{\max})$. Moreover, $(\tilde{\Phi}(\theta^{\max}), \{\tilde{\Phi}(\theta_i)\}_{i \neq k})$ and $(\tilde{\Phi}(\theta^{\min}), \{\tilde{\Phi}(\theta_i)\}_{i \neq k})$ are in $SC(\tilde{X})$. Therefore, if we define $\bar{\theta}_k$ as $\theta^{\max}$ and $\hat{\theta}_k$ as $\theta^{\min}$, we have a contradiction with $(\star)$.

On the other hand, if $|\Theta^I(\tilde{\succ})| = 1$, it would be trivial to find an individual and a pair of alternative types for this agent, such that the new profiles are still in $SC(\tilde{X})$. Let us, therefore, consider the possibility that $|\Theta^I(\tilde{\succ})| = 2$, i.e. that $\Theta^I(\tilde{\succ}) = \{\theta^1, \theta^2\}$. It is obvious that $\tilde{\Phi}(\theta^1)$ and $\tilde{\Phi}(\theta^2)$ differ in at least a pair of alternatives, say $w > z$. Then, we define $\bar{\theta}$ such that $\tilde{\Phi}(\bar{\theta})$ coincides with $\tilde{\Phi}(\theta^1)$ for every pair of alternatives, except for $z$ and $w$, and set $w \, \tilde{\Phi}(\bar{\theta}) \, z$ if and only if $w \, \tilde{\Phi}(\theta^2) \, z$. If $\tilde{\Phi}(\bar{\theta}) \neq \tilde{\Phi}(\theta^2)$, $\bar{\theta}$ and $\hat{\theta} = \theta^2$ constitutes a pair of alternative types for an agent of type $\theta^1$ that violates $(\star)$. Otherwise, if $\tilde{\Phi}(\bar{\theta}) = \tilde{\Phi}(\theta^2)$, just consider any pair of elements $x, y \in \tilde{X}$, $x > y$ with $x \neq w$ or $y \neq z$ (which exists since $|\tilde{X}| > 2$) for which $x \, \tilde{\Phi}(\theta^2) \, y$. Define $\bar{\theta}'$ such that $y \, \tilde{\Phi}(\bar{\theta}') \, x$. Then $\bar{\theta}'$ and $\hat{\theta} = \theta^1$ constitutes a pair of alternative types for an agent of type $\theta^2$ that, again, violates $(\star)$. If such a pair $\{x, y\}$ does not exists, then $\tilde{\Phi}(\theta^1)$ must be such that $x_1 \, \tilde{\Phi}(\theta^1) \, x_2 \, \tilde{\Phi}(\theta^1) \, x_3 \, \ldots \, x_{s-1} \, \tilde{\Phi}(\theta^1) \, x_s$.[13] But then there must exist a

[13]Remember that the set $\tilde{X}$ is such that $x_j \leq x_k$ for $j \leq k$

pair $x, y \in \tilde{X}$, say $x > y$, and a type $\bar{\theta}'' \in \Theta$ such that $\tilde{\Phi}(\bar{\theta}'')$ coincides with $\tilde{\Phi}(\theta^2)$, but $y\,\tilde{\Phi}(\theta^2)\,x$ while $x\,\tilde{\Phi}(\bar{\theta}'')\,y$ and $(\tilde{\Phi}(\bar{\theta}''_k), \{\tilde{\Phi}(\theta_i\}_{i\neq k})) \in SC(\tilde{X})$, where $k$ is assumed to be an agent of type $\theta^2$. Again, agent $k$ and the pair $\bar{\theta}''$ and $\hat{\theta} = \theta^1$ contradicts $(\star)$. 2

**Lemma 2** *For any efficient and tops-only social choice rule* $f : SC(\tilde{X}) \to \tilde{X}$ *that satisfies* $(**)$, $\exists\, k \in I$ *and* $\hat{\succ}_k$ *and* $\bar{\succ}_k$, *such that they verify* simultaneously *Properties 1-4 in the text.*

Proof Suppose, to the contrary, that for every $k$ and every pair of individual preferences $\hat{\succ}_k$, $\bar{\succ}_k$ either:

$\mathbf{P_1}$: At least one of $(\hat{\succ}_k, \tilde{\succ}_{-k})$ or $(\bar{\succ}_k, \tilde{\succ}_{-k})$ is not in $SC(\tilde{X})$; or,
$\mathbf{P_2}$: $f(\hat{\succ}_k, \tilde{\succ}_{-k}) = x$ ; or,
$\mathbf{P_3}$: $\tau(\bar{\succ}_k) \neq \tau(\tilde{\succ}_k)$; or,
$\mathbf{P_4}$: Either $f(\tilde{\succ}) = f(\hat{\succ}_k, \tilde{\succ}_{-k})$ or $f(\tilde{\succ})\,\bar{\succ}_k\,f(\hat{\succ}_k, \tilde{\succ}_{-k})$.

Let us consider each possibility in order to get to a contradiction.

$\mathbf{P_1}$: This leads to a contradiction with Lemma 1.

$\mathbf{P_2}$: Suppose that for every $k$ and every $\hat{\succ}_k$, such that $(\hat{\succ}_k, \tilde{\succ}_{-k}) \in SC(\tilde{X})$, $f(\hat{\succ}_k, \tilde{\succ}_{-k}) = x$. In words, this means that no individual deviation from $\tilde{\succ} = (\tilde{\succ}_1, \ldots, \tilde{\succ}_n)$ matters. This implies by induction that $f(\hat{\succ}_{I'}, \tilde{\succ}_{I''}) = f(\tilde{\succ})$, where $(I', I'')$ is any partition of the set of agents. In effect, the base case, where $I' = \{k\}$ and $I'' = I \setminus \{k\}$ is in fact our hypothesis: $f(\hat{\succ}_k, \tilde{\succ}_{-k}) = f(\tilde{\succ})$. Now, suppose that $f(\hat{\succ}_{I'}, \tilde{\succ}_{I''}) = f(\tilde{\succ})$ where $k \in I''$. Then, by transitivity, $f(\hat{\succ}_{I'}, \tilde{\succ}_{I''}) = f(\hat{\succ}_k, \tilde{\succ}_{-k})$. We will show that $f(\hat{\succ}_{I'+k}, \tilde{\succ}_{I''-k}) = f(\tilde{\succ})$. Suppose, to the contrary, that after a new deviation, $f(\hat{\succ}_{I'+k}, \tilde{\succ}_{I''-k}) \neq f(\tilde{\succ})$, i.e. that $f(\hat{\succ}_{I'+k}, \tilde{\succ}_{I''-k}) \neq f(\hat{\succ}_{I'}, \tilde{\succ}_{I''})$. Without loss of generality, assume that $f(\hat{\succ}_{I'}, \tilde{\succ}_{I''}) < f(\hat{\succ}_{I'+k}, \tilde{\succ}_{I''-k})$. Suppose $k$ and $\hat{\succ}_k$ are such that either $f(\hat{\succ}_{I'+k}, \tilde{\succ}_{I''-k})\ \tilde{\succ}_k\ f(\hat{\succ}_{I'}, \tilde{\succ}_{I''})$ or $f(\hat{\succ}_{I'}, \tilde{\succ}_{I''})\ \hat{\succ}_k\ f(\hat{\succ}_{I'+k}, \tilde{\succ}_{I''-k})$. Notice that a $k$ verifying this may exist, since $\hat{\succ}_k$ and $\tilde{\succ}_k$ should differ in the valuation of at least a pair of points. Then, it is immediate to see that either case implies a violation of strategy-proofness. On the contrary, assume that for every $k$ and every $\hat{\succ}_k$ both $f(\hat{\succ}_{I'}, \tilde{\succ}_{I''})\ \tilde{\succ}_k\ f(\hat{\succ}_{I'+k}, \tilde{\succ}_{I''-k})$ and $f(\hat{\succ}_{I'+k}, \tilde{\succ}_{I''-k})\ \hat{\succ}_k\ f(\hat{\succ}_{I'}, \tilde{\succ}_{I''})$. It is easy to find a $k$ and an ordering $\hat{\succ}_k$ such that these conditions are not verified simultaneously. Therefore, we have proved our inductive hypothesis, i.e. that $f(\hat{\succ}_{I'+k}, \tilde{\succ}_{I''-k}) = f(\tilde{\succ})$ for every partition $(I', I'')$ of the set of agents and every $k \in I''$. In the limit, we have that $f(\hat{\succ}) = f(\tilde{\succ})$. Once achieved this limit, consider the case in which preferences are identical for all agents. Concretely, take the profile $\hat{\succ}$. Define the permutation $\sigma : I \to I$ such that, for every $i, l \in I$, $\sigma(i) = \sigma_i < \sigma_l = \sigma(l)$ if $\theta_i < \theta_l$; and, if $\theta_i = \theta_l$ and $l < i$, set $\sigma(l) > \sigma(i)$. To avoid to work explicitly with the permutation, in what follows there is no confusion in supposing that the index of each individual refers to his new number under the permutation. Now choose sequentially $\hat{\succ}_k = \tilde{\succ}_{i^*}$ for each agent $k = i^* + 1, i^* + 2, \ldots, n^*$ and then for $k = i^* - 1, i^* - 2, \ldots, 1^*$. (Remember that we were considering the $i$-th positional dictator choice rule). By $(**)$, $f(\tilde{\succ}) \neq \tau(\tilde{\succ}_{i^*})$. Therefore, $\tau(\tilde{\succ}_{i^*})\,\hat{\succ}_k\,f(\hat{\succ})$ for every $k \in I$, contradicting the fact that $f$ is Pareto efficient.

**$\mathbf{P_3}$**: Suppose, by contradiction, that for every $k$ and every $\bar{\succ}_k$, such that $(\bar{\succ}_k, \tilde{\succ}_{-k}) \in SC(\tilde{X})$, $\tau(\bar{\succ}_k) \neq \tau(\tilde{\succ}_k)$. It is immediate to see that such statement is false. Just consider the profile $(\tilde{\succ}_1, \ldots, \tilde{\succ}_n)$ and the type $\theta^{min} = \min_{(\theta)}\ \Theta^I(\tilde{\succ})$. Then, for an individual $i$ of type $\theta^{min}$, it is always possible to define a preference relation $\bar{\succ}_i$ such that $(\bar{\succ}_i, \tilde{\succ}_{-i}) \in SC(\tilde{X})$, $\tau(\bar{\succ}_i) = \tau(\tilde{\succ}_i)$, and $\bar{\succ}_i$ and $\tilde{\succ}_i$ differ in the ranking of at least a pair of distinct alternatives $x, y \in \tilde{X} \backslash \{\tau(\bar{\succ}_i), \tau(\tilde{\succ}_i)\}$.

**$\mathbf{P_4}$**: Suppose, by contradiction, that for every $k$ and every two preference orderings $\hat{\succ}_k$ and $\bar{\succ}_k$, such that $(\hat{\succ}_k, \tilde{\succ}_{-k}), (\bar{\succ}_k, \tilde{\succ}_{-k}) \in SC(\tilde{X})$, either $f(\tilde{\succ}) = f(\hat{\succ}_k, \tilde{\succ}_{-k})$ or $f(\tilde{\succ})\,\bar{\succ}_k\,f(\hat{\succ}_k, \tilde{\succ}_{-k})$. Since we have already contradicted $\mathbf{P_2}$, the first possibility is ruled out. Thus, without loss of generality, assume $f(\tilde{\succ}) < f(\hat{\succ}_k, \tilde{\succ}_{-k})$. By single-crossing, $f(\tilde{\succ})\,\bar{\succ}_k\,f(\hat{\succ}_k, \tilde{\succ}_{-k})$ $\Rightarrow f(\tilde{\succ})\,\tilde{\succ}_i\,f(\hat{\succ}_k, \tilde{\succ}_{-k})$ for every $i$ such that $\theta_i \leq \bar{\theta}_k$, where $\theta_i$ and $\bar{\theta}_k$ are such that $\tilde{\Phi}(\theta_i) = \tilde{\succ}_i$ and $\tilde{\Phi}(\bar{\theta}_k) = \bar{\succ}_k$, respectively. On the other hand, $f(\hat{\succ}_k, \tilde{\succ}_{-k}) \hat{\succ}_k f(\tilde{\succ})$. Otherwise, $k$ can manipulate $f$ at $(\hat{\succ}_k, \tilde{\succ}_{-k})$ via $\tilde{\succ}_k$. Combining this with the previous claim, it follows that $\hat{\theta}_k > \bar{\theta}_k$. But then there must exist a type $\bar{\theta}'_k \in \Theta$, $\bar{\theta}_k < \bar{\theta}'_k < \hat{\theta}_k$, such that the associated ordering $\tilde{\Phi}(\bar{\theta}'_k) = \bar{\succ}'_k$ coincides with $\bar{\succ}_k$ except in the ranking of the alternatives $f(\tilde{\succ})$ and $f(\hat{\succ}_k, \tilde{\succ}_{-k})$. That is, $f(\hat{\succ}_k, \tilde{\succ}_{-k}) \bar{\succ}'_k f(\tilde{\succ})$. Thus, the pair $\hat{\succ}_k$ and $\bar{\succ}'_k$ contradicts $\mathbf{P_4}$.

Thus, since we disproved $\mathbf{P_1} - \mathbf{P_4}$, there must exist a $k$ with two alternative preferences, $\hat{\succ}_k$ and $\bar{\succ}_k$ that verifies simultaneously Properties $1-4$. 2

**Lemma 3** *For any tops-only social choice rule $f : SC(\tilde{X}) \rightarrow \tilde{X}$ that satisfies* $(**)$ *and $A_f > 2$, $\exists\, k \in I$ and $\hat{\succ}_k$ and $\bar{\succ}_k$, such that they verify* simultaneously *Properties 1-4 in the text.*

Proof Suppose, as in the proof of Lemma 2, that for every $k$ and every pair of individual preferences $\hat{\succ}_k$, $\bar{\succ}_k$ either:

$\mathbf{P_1}$: At least one of $(\hat{\succ}_k, \tilde{\succ}_{-k})$ or $(\bar{\succ}_k, \tilde{\succ}_{-k})$ is not in $SC(\tilde{X})$; or,
$\mathbf{P'_2}$: $f(\hat{\succ}_k, \tilde{\succ}_{-k}) = x$ ; or,
$\mathbf{P_3}$: $\tau(\bar{\succ}_k) \neq \tau(\tilde{\succ}_k)$; or,
$\mathbf{P_4}$: Either $f(\tilde{\succ}) = f(\hat{\succ}_k, \tilde{\succ}_{-k})$ or $f(\tilde{\succ}) \bar{\succ}_k f(\hat{\succ}_k, \tilde{\succ}_{-k})$.

The arguments in the proof of Lemma 2 can be repeated to show that $\mathbf{P_1}$, $\mathbf{P_3}$ and $\mathbf{P_4}$ are false. To show that $\mathbf{P'_2}$ is also false consider the following argument:

$\mathbf{P'_2}$: For expositional simplicity, consider the *leftist choice rule*, instead of the $i$-th positional dictator choice rule. That is, take the combination of $\alpha$s in $\{0, +\infty\}^{n-1}$ that always chooses the smallest reported peak of a real voter. Suppose the profile $(\tilde{\succ}_1, \ldots, \tilde{\succ}_n) \in SC(\tilde{X})$ is such that $f(\tilde{\succ}) = x$, while $x \neq \tau(\tilde{\succ}_i)$, where $i$ is the agent which has the first ranked peak in the distribution $\tau(\tilde{\succ}_1), \ldots, \tau(\tilde{\succ}_n)$. After renaming the agents as in $\mathbf{P_2}$, assume that, for every $k$ and every $\hat{\succ}_k$, such that $(\hat{\succ}_k, \tilde{\succ}_{-k}) \in SC(\tilde{X})$, $f(\hat{\succ}_k, \tilde{\succ}_{-k}) = f(\tilde{\succ}) = x$. By the same reasoning applied in $\mathbf{P_2}$, it follows that $f(\hat{\succ}_{I'_\sigma}, \tilde{\succ}_{I''_\sigma}) = f(\tilde{\succ})$ for every partition $(I'_\sigma, I''_\sigma)$ of the set of agents. On the other hand, since $|A_f| > 2$, there exist $y \in \tilde{X}$, $y \neq x$, and $(\tilde{\succ}'_1, \ldots, \tilde{\succ}'_n) \in SC(\tilde{X})$ such that $f(\tilde{\succ}'_1, \ldots, \tilde{\succ}'_n) = y$. We want to prove that, after a finite number of individual deviations from $(\tilde{\succ}_1, \ldots, \tilde{\succ}_n)$, we can achieve a profile $(\hat{\succ}_1, \ldots, \hat{\succ}_n) \in SC(\tilde{X})$ such that $\tau(\hat{\succ}_i) = \tau(\tilde{\succ}'_i)$ $\forall i \in I_\sigma$, while $f(\hat{\succ}) \neq f(\tilde{\succ}')$. To do that, consider the most leftist type $\theta^1 \in \Theta$, characterized by the binary relation $x_1 \, \tilde{\Phi}(\theta^1) \, x_2 \ldots x_{s-1} \, \tilde{\Phi}(\theta^1) \, x_s$. Then, by sequentially deviating each agent $k = 1, \ldots, n$ from $\tilde{\succ}_k$ to $\tilde{\Phi}(\theta^1)$, we obtain the unanimous profile $(\tilde{\Phi}(\theta^1), \ldots, \tilde{\Phi}(\theta^1))$, which is obviously in $SC(\tilde{X})$. Moreover, $f(\tilde{\Phi}(\theta^1), \ldots, \tilde{\Phi}(\theta^1)) = x$. But now we can take individual in the $n$-th position and define a sequence of deviations for this agent $\hat{\succ}^1_n, \ldots, \hat{\succ}^h_n$, where $\hat{\succ}^1_n$ is obtained from $\tilde{\Phi}(\theta^1)$ by moving up to the first position (to the top) the greatest alternative in $T(\tilde{X}, \tilde{\succ}')$; $\hat{\succ}^2_n$ is obtained from $\hat{\succ}^1_n$ by moving up to the second position the second higher alternative in $T(\tilde{X}, \tilde{\succ}')$; etc. Clearly, the profile $(\tilde{\Phi}(\theta^1), \ldots, \tilde{\Phi}(\theta^1), \tilde{\Phi}(\hat{\theta}^j_n)) \in SC(\tilde{X})$, for each $j = 1, \ldots, h$, where $\tilde{\Phi}(\hat{\theta}^j_n) = \hat{\succ}^j_n$. Moreover, $f(\tilde{\Phi}(\theta^1), \ldots, \tilde{\Phi}(\theta^1), \tilde{\Phi}(\hat{\theta}^j_n)) = x$. Denote $\hat{\succ}^h_n = \hat{\succ}_n$. Consider now the individual in the $(n-1)$-th position. Define $\hat{\succ}^1_{n-1}, \ldots, \hat{\succ}^{h-1}_{n-1}$, where $\hat{\succ}^1_{n-1}$ is obtained from $\tilde{\Phi}(\theta^1)$ by moving up to the first position (to the top) the second greatest alternative in $T(\tilde{X}, \tilde{\succ}')$; $\hat{\succ}^2_{n-1}$ is obtained from $\hat{\succ}^1_{n-1}$ by moving up to the second position the third higher alternative in $T(\tilde{X}, \tilde{\succ}')$; etc. After repeating this process for each agent, we finally reach individual in the 1-st for which we simply define an alternative ordering $\hat{\succ}_1$ that moves up to the top the smallest element in $T(\tilde{X}, \tilde{\succ}')$. Thus, by proceeding in this way, we derive a profile $(\tilde{\Phi}(\hat{\theta}_1), \ldots, \tilde{\Phi}(\hat{\theta}_n)) \in SC(\tilde{X})$ such that $\tau(\tilde{\Phi}(\hat{\theta}_i)) = \tau(\tilde{\succ}'_i)$, but $f(\hat{\succ}_1, \ldots, \hat{\succ}_n) = x \neq f(\tilde{\succ}'_1, \ldots, \tilde{\succ}'_n)$, contradicting the tops-only condition. Thus $\mathbf{P}'_{\mathbf{2}}$ is also false.[14] 2

**Lemma 4** *For any social choice rule $f : SC(\tilde{X}) \to \tilde{X}$ that satisfies* $(**)$ *and $|A_f| > 2$, $\exists\, k \in I$ and $\hat{\succ}_k$ and $\tilde{\succ}_k$, such that they verify* simultaneously *Properties 1-4 in the text.*

[14]Notice that, to get to a contradiction, it is crucial to choose $x \neq \tau(\succ_1)$. Otherwise, agent 1 can manipulate $f$ at $(\hat{\succ}_1, \ldots, \hat{\succ}_n)$ via $\tilde{\Phi}(\theta^1)$, violating the strategy-proofness of $f$.

Proof Suppose, as in the proof of Lemma 2 and 3, that for every $k$ and every pair of individual preferences $\hat{\succ}_k$, $\bar{\succ}_k$ either:

$\mathbf{P_1}$: At least one of $(\hat{\succ}_k, \tilde{\succ}_{-k})$ or $(\bar{\succ}_k, \tilde{\succ}_{-k})$ is not in $SC(\tilde{X})$; or,
$\mathbf{P_2''}$: $f(\hat{\succ}_k, \tilde{\succ}_{-k}) = x$ ; or,
$\mathbf{P_3}$: $\tau(\bar{\succ}_k) \neq \tau(\tilde{\succ}_k)$; or,
$\mathbf{P_4}$: Either $f(\tilde{\succ}) = f(\hat{\succ}_k, \tilde{\succ}_{-k})$ or $f(\tilde{\succ}) \mathrel{\bar{\succ}_k} f(\hat{\succ}_k, \tilde{\succ}_{-k})$.

Following the same arguments that before, it is immediate to prove that $\mathbf{P_1}$, $\mathbf{P_3}$ and $\mathbf{P_4}$ are false. To show that $\mathbf{P_2''}$ is also false consider again the argument for $\mathbf{P_2}$ in the proof of Lemma 2. Recall that we have a profile $(\hat{\succ}_{\sigma_1}, \ldots, \hat{\succ}_{\sigma_n})$ that obtains from $\tilde{\succ}$ by means of a sequence of deviations, such that $f(\hat{\succ}_{\sigma_1}, \ldots, \hat{\succ}_{\sigma_n}) = f(\tilde{\succ})$. In particular, this profile can be such that $\hat{\succ}_{\sigma_j} = \hat{\succ}_{\sigma_l}$ for every pair $j, l \in I$. Moreover, it is possible to choose, for each $\sigma_i \in I_\sigma$, $\tau(\hat{\succ}_{\sigma_i}) = y \neq x$, since $|A_f| > 2$. But then we have, on one hand that $f(\hat{\succ}_{\sigma_1}, \ldots, \hat{\succ}_{\sigma_n}) = x$ while on the other, $f(\hat{\succ}_{\sigma_1}, \ldots, \hat{\succ}_{\sigma_n}) = y$ (by Proposition 2). Contradiction. Then, $\mathbf{P_2''}$ is also false. 2