THE PENALTY-KICK GAME UNDER INCOMPLETE INFORMATION

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Abstract

This paper presents a model of the penalty-kick game between a soccer goalkeeper and a kicker in which there is uncertainty about the kicker's type (and there are two possible types of kicker). To find a solution for this game we use the concept of Bayesian equilibrium, and we find that, typically, one the kicker's types will play a mixed strategy while the other type will choose a pure strategy. Comparing this equilibrium with the corresponding Nash equilibria under complete information, we find that the expected scoring probability increases (so that, on average, the goalkeeper is worse off).

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1. Introduction

The penalty-kick game, in which a soccer goalkeeper and a kicker face each other, has become an important example in the game-theory literature to analyze mixed-strategy Nash equilibria. The reason of this importance probably has to do with the fact that it is a game whose solution generates a clear theoretical prediction and, at the same time, it is relatively easy to gather data about actual outcomes of the game. Besides, this is a game in which it is not necessary to perform laboratory experiments, since penalty-kick situations in soccer matches are frequent, and soccer players are usually trained to shoot and to save penalty kicks. Moreover, there exist relatively large records of the different details involved in many actual penalty kicks shot at various soccer leagues (e.g., if they were scored or not, the side chosen by the goalkeeper and the kicker, who were the goalkeeper and the kicker, the situation of the match at the moment of the shot, etc.), and this helps to control for several factors that may influence the result of the game.

All the game-theoretic literature that we know about penalty kicks analyzes this situation as a game of complete information, i.e., as a game in which the two players know the characteristics of each other, and hence they know the expected payoffs that they will receive in the different strategy profiles of the game. There is a good reason for this assumption, which is the idea that goalkeepers and kickers in a professional soccer league are usually well-known players whose main characteristics are recognized by their opponents, and those characteristics are precisely the ones that define the parameters which establish the expected payoff of the penalty-kick game. Complete-information games, moreover, are also easier to solve and, perhaps more importantly, are easier to test empirically. This ease is probably the best explanation for the success of the penalty-kick game as a prominent example in the game-theoretic literature.

Not all soccer penalty kicks, however, are shot in situations in which it is reasonable to assume complete information. In many cases, especially in amateur matches and in matches between teams that belong to different leagues, it is possible that players do not know each other and, therefore, are uncertain about several important characteristics that may influence the outcome of the game. It is also possible that the kicker (and, less usually, the goalkeeper) is a player who is not the “typical choice” in his team, because he is out of the match or because the team has decided to change him due to a poor past performance. Moreover, as penalty kicks are sometimes used as tie-breakers in some tournaments, and this requires that several players from each team shoot penalty kicks, then it is possible that some of the designated kickers do not usually shoot penalty kicks in professional matches. This may generate a situation in which the goalkeeper is uncertain about some of the kicker's relevant characteristics, changing the game into one with incomplete information.

When we have to analyze a game with incomplete information, the main solution concept for games with complete information (i.e., Nash equilibrium) is usually not available. Since the seminal contribution by Harsanyi (1967), however, we have an alternative concept to apply in these cases, which is the so-called "Bayesian equilibrium". This equilibrium relies on the idea that, under incomplete information, players typically have data about the probabilities of their opponents' characteristics, and this allows them to figure out which are the different "types" of opponents that they may face and the probability associated to each type. With that information we can build an equilibrium in which each player's type plays his best response to their opponents' strategies, taking into account the probability of facing each opponent's type.

In this paper we will develop a model of a penalty-kick game in which there is a single type of goalkeeper and two types of kickers. This is consistent with several results that appear in the literature, and especially with the empirical observation of Chiappori, Levitt and Groseclose (2002) that professional goalkeepers are basically homogeneous in their characteristics as penalty-kick savers, and that the main variation that we observe comes from the kickers' side. In their contribution, for example, they find that the complete-information penalty-kick game can have two different types of Nash equilibria, depending on the scoring probability associated to shooting to the center of the goal. If that probability is relatively low, then the equilibrium is what they call a "restricted-randomization equilibrium", in which both the goalkeeper and the kicker randomize between left and right, but they never choose the center. If conversely, the scoring probability of shooting center (when the goalkeeper chooses one of his sides) is relatively large, then we find a "general-randomization equilibrium" (in which both the goalkeeper and the kicker randomize among left, right and center).

In a recent paper by Jabbour and Minquet (2009), the authors allow for an additional strategy dimension which is the height of the kicker's shot, and assume that shooting high to one of the sides (left or right) assures the kicker a certain scoring probability (because the goalkeeper cannot save the shot, and his only hope is that the kicker shoots outside the goal). In this case we also have two types of Nash equilibria, which depend on the scoring probability of shooting high: if this probability is relatively small, then the kicker will randomize between shooting right-low or left-low (but he will never shoot high); if it is large, then the kicker will strictly prefer to shoot high, and he will always choose his “natural side” (i.e., the goalkeeper’s right, if the kicker is right-footed, or the goalkeeper’s left, if the kicker is left-footed).

Different types of kickers also change the Nash equilibrium in the simplest versions of the penalty-kick game. In Palacios-Huerta (2003), for example, both the goalkeeper and the kicker choose between two strategies (left and right), but the equilibrium mixed strategies are functions of the scoring probabilities of the four possible strategy profiles. Changing one of these parameters, therefore, changes the equilibrium; and playing a strategy that is an equilibrium one for a certain set of parameters when the set of parameters is different, consequently, implies that the other player’s best response is a pure strategy and not a mixed one. This last situation, of course, is never a Nash equilibrium in a complete information setting, but it may well be part of a Bayesian equilibrium if we allow for incomplete information about the kicker’s characteristics.

In the next section of this paper we will present a simple model in which we will allow for uncertainty about one of the four parameters that define the scoring probabilities of the 2x2 version of the penalty-kick game. This change, together with the inclusion of an additional parameter that defines the probability distribution of the kicker’s types, will generate a new game in which the kicker plays knowing the goalkeeper’s characteristics but the goalkeeper plays against an uncertain opponent (who may be of two different types). We will also compare the solution of this game with their complete-information counterparts, i.e., with the Nash equilibria of the games in which the goalkeeper alternatively faces each of the kicker’s types, knowing who his opponent is.

In the third section of the paper, we will develop a few theoretical results that arise from our model, that refer to the differences in expected scoring probabilities between the two possible complete-information games and between those games and the incomplete-information version. The fourth section will include a numerical example of the model, and the fifth section will be devoted to the conclusions and to some final remarks.

2. The model

Following the notation that appears in Coloma (2007), we will build a game in which the kicker has to choose between his natural side (the goalkeeper’s right, if the kicker is right-footed, or the goalkeeper’s left, if the kicker is left-footed) and his opposite side (the goalkeeper's left, if the kicker is right-footed, or the goalkeeper's right, if the kicker is left-footed). Similarly, the goalkeeper has to choose between the kicker's natural side (NS) and the kicker's opposite side (OS). The probability of scoring if both the kicker and the goalkeeper choose NS is P_N, while the probability of scoring if both the kicker and the goalkeeper choose OS is P_O. If the kicker chooses NS but the goalkeeper chooses OS, then the scoring probability is π_N, while the scoring probability in the case that the kicker chooses OS and the goalkeeper chooses NS is π_O. As this is a constant-sum game in which the kicker wins if he scores and the goalkeeper wins if the kicker does not score, then the kicker's expected payoff can be associated to the scoring probability and the goalkeeper's expected payoff can be associated to the complement of that probability. As it is a simultaneous game, then the kicker's strategy space consists of two strategies (NS and OS) and the goalkeeper strategy space also consists of two strategies (NS and OS).

Both the theoretical and the empirical literature agree that the scoring probabilities of the penalty-kick game have to be defined so as "$\pi_N \geq \pi_O > P_N > P_O$", and these conditions guarantee that the Nash equilibrium of the complete-information version of the game is a mixed-strategy one, in which the kicker chooses NS with a certain probability n (and chooses OS with probability $1\text{-}n$), and the goalkeeper chooses NS with a certain probability v (and chooses OS with probability $1\text{-}v$). Moreover, the fact that "$P_N > P_O$" makes that, in equilibrium, both n and v are greater than ½. Besides, if π_N is strictly greater than π_O, then it will also hold that, in equilibrium, "$v > n$", while if it holds that "$\pi_N = \pi_O$", then equilibrium implies that "$n = v$"[1].

Table 1: Scoring-probability matrix

		Goalkeeper	
		NS	OS
Kicker 1 (Prob θ)	NS	P_N	π_N
	OS	π_O	P_O
Kicker 2 (Prob 1-θ)	NS	P_N	π_O
	OS	π_O	P_O

One of the easiest ways of building a penalty-kick game with incomplete information is to assume that there is a single type of goalkeeper and two types of kickers. Kicker 1 is someone for whom "$\pi_N > \pi_O$", while kicker 2 is someone for whom "$\pi_N = \pi_O$". To simplify

[1] The first of these results was first presented by Chiappori, Levitt and Groseclose (2002), while the second one appears in Baumann, Friehe and Wedow (2011).

matters even further, we will assume that the values of "P_N" and "P_O" are the same for both types of kickers, and that "π_O" is also the same for both types. Then our model depends on the standard four parameters (π_N , π_O, P_N and P_O), plus an additional parameter θ that represents the probability that the goalkeeper faces kicker 1 (while the probability of facing kicker 2 is $1-\theta$). The complete probability matrix is the one that appears on table 1.

If we first consider the case in which "$\pi_N > \pi_O$" (game 1), then the complete-information Nash equilibrium implies that both the goalkeeper and the kicker are indifferent between choosing NS and OS. For this to happen, it should hold that:

$$P_N \cdot n_1 + \pi_O \cdot (1-n_1) = \pi_N \cdot n_1 + P_O \cdot (1-n_1) \quad \Rightarrow \quad n_1 = \frac{\pi_O - P_O}{\pi_N + \pi_O - P_N - P_O} \quad (1);$$

$$P_N \cdot \nu_1 + \pi_N \cdot (1-\nu_1) = \pi_O \cdot \nu_1 + P_O \cdot (1-\nu_1) \quad \Rightarrow \quad \nu_1 = \frac{\pi_N - P_O}{\pi_N + \pi_O - P_N - P_O} \quad (2).$$

Alternatively, if "$\pi_N = \pi_O$" (game 2), then the complete-information Nash equilibrium solution of the game occurs when:

$$P_N \cdot n_2 + \pi_O \cdot (1-n_2) = \pi_O \cdot n_2 + P_O \cdot (1-n_2) \quad \Rightarrow \quad n_2 = \frac{\pi_O - P_O}{2 \cdot \pi_O - P_N - P_O} \quad (3);$$

$$P_N \cdot \nu_2 + \pi_O \cdot (1-\nu_2) = \pi_O \cdot \nu_2 + P_O \cdot (1-\nu_2) \quad \Rightarrow \quad \nu_2 = \frac{\pi_O - P_O}{2 \cdot \pi_O - P_N - P_O} \quad (4).$$

The equilibrium values of n_1, n_2, v_1 and v_2 can be compared to obtain some relationships that will later be useful to analyze the Bayesian equilibrium of the corresponding incomplete-information game. By simple observation we find that "$n_2 = v_2$" (as we have already anticipated) and that "$n_2 > n_1$" (since both equilibrium expressions have the same numerator but n_1's denominator is greater than n_2's). We can also prove that "$v_1 > v_2$", as the following lemma shows.

Lemma 1: *Under complete information, the Nash equilibrium solution of the penalty-kick game implies that* "$v_1 > v_2$".

Proof: Suppose instead that "$v_1 < v_2$". Then it should hold that:

$$\nu_1 = \frac{\pi_N - P_O}{\pi_N + \pi_O - P_N - P_O} < \frac{\pi_O - P_O}{2 \cdot \pi_O - P_N - P_O} = \nu_2 \quad .$$

But if this is so, then it should also hold that:

$$(\pi_N - P_O) \cdot (2 \cdot \pi_O - P_N - P_O) < (\pi_O - P_O) \cdot (\pi_N + \pi_O - P_N - P_O) \quad ;$$

which implies that:

$$(2\pi_O - P_N)\pi_N - (2\pi_O + \pi_N)P_O + P_O(P_N + P_O) < (\pi_N + \pi_O - P_N)\pi_O - (2\pi_O + \pi_N)P_O + P_O(P_N + P_O) \quad ;$$

$$(2 \cdot \pi_O - P_N) \cdot \pi_N < (\pi_N + \pi_O - P_N) \cdot \pi_O \qquad \Rightarrow \qquad (\pi_O - P_N) \cdot \pi_N < (\pi_O - P_N) \cdot \pi_O$$

$$\Rightarrow \qquad (\pi_O - P_N) \cdot (\pi_N - \pi_O) < 0 .$$

But, as we know that "$\pi_O > P_N$" and "$\pi_N > \pi_O$", then this is a contradiction. Therefore it holds that "$v_1 > v_2$", q.e.d.

Let us now turn to the incomplete-information case, in which the goalkeeper does not know if he is facing kicker 1 or kicker 2, but the kicker knows his type (and also the unique goalkeeper's type). In this case the goalkeeper will choose NS with some probability v_M, regardless of the fact that he is facing kicker 1 or kicker 2. Given this, kicker 1 is strictly better off by shooting NS if it holds that "$v_M < v_1$", while he is strictly better off by shooting OS if it holds that "$v_M > v_1$". Correspondingly, kicker 2 is strictly better off by shooting NS if it holds that "$v_M < v_2$", while he is strictly better off by shooting OS if it holds that "$v_M > v_2$".

Let us first assume that, as "$v_1 > v_2$", then the goalkeeper chooses a value for v_M such that "$v_1 > v_M > v_2$". In this case kicker 1's best response will be to play NS as a pure strategy, and kicker 2's best response will be to play OS. But this could only be an equilibrium if the goalkeeper is indifferent between choosing NS and OS himself, for which it should hold that:

$$P_N \cdot \theta + \pi_O \cdot (1-\theta) = \pi_N \cdot \theta + P_O \cdot (1-\theta) \qquad \Rightarrow \qquad \theta = \frac{\pi_O - P_O}{\pi_N + \pi_O - P_N - P_O} \qquad (5) \ ;$$

and this is something that will generically occur with zero probability[2]. We should therefore look for alternative types of equilibria in which one of the kicker's types plays a pure strategy and the other one plays a mixed strategy. Two of those equilibria exist, and we will label them "case A" and "case B".

In case A, kicker 1 chooses NS, and both kicker 2 and the goalkeeper play mixed strategies. For this to occur, v_M has to be equal to v_2, and therefore kicker 1 is strictly better off by playing NS and kicker 2 is indifferent between NS and OS. For the goalkeeper to be indifferent between NS and OS, however, we need that:

$$P_N \cdot \theta + [P_N \cdot n_2 + \pi_O \cdot (1-n_2)] \cdot (1-\theta) = \pi_N \cdot \theta + [\pi_O \cdot n_2 + P_O \cdot (1-n_2)] \cdot (1-\theta)$$

$$\Rightarrow \qquad n_2 = \frac{(\pi_O - P_O) - (\pi_N - P_N) \cdot \theta / (1-\theta)}{2 \cdot \pi_O - P_N - P_O} \qquad (6) \ .$$

In case B, conversely, kicker 2 always chooses OS, and both kicker 1 and the

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[2] This is because, as θ could be any real number between zero and one, then the probability that it is exactly

goalkeeper play mixed strategies. For this to occur, v_M has to be equal to v_1, and therefore kicker 2 is strictly better off by playing OS and kicker 1 is indifferent between NS and OS. For the goalkeeper to be indifferent between NS and OS, however, we need that:

$\pi_O\cdot(1-\theta) + [P_N\cdot n_1 + \pi_O\cdot(1-n_1)]\cdot\theta = P_O\cdot(1-\theta) + [\pi_N\cdot n_1 + P_O\cdot(1-n_1)]\cdot\theta$

$$\Rightarrow \quad n_1 = \frac{(\pi_O - P_O)/\theta}{\pi_N + \pi_O - P_N - P_O} \quad (7) .$$

Both equilibria under cases A and B can also be seen as situations in which the goalkeeper is randomizing between NS and OS because he has the belief that one of the players is choosing a pure strategy with probability one, and the other player is playing a mixed strategy such as the one described by equations 6 or 7. Under case A, therefore, his belief is that, on average, the kicker will choose NS with a certain probability n_A equal to:

$$n_A = \theta\cdot n_1 + (1-\theta)\cdot n_2 = \theta + (1-\theta)\frac{(\pi_O - P_O) - (\pi_N - P_N)\theta/(1-\theta)}{2\cdot\pi_O - P_N - P_O} = \frac{(\pi_O - P_O) - (\pi_N - P_N)\theta}{2\cdot\pi_O - P_N - P_O} \quad (8) ;$$

while under case B his belief is that, on average, the kicker will choose NS with a certain probability n_B equal to:

$$n_B = \theta\cdot n_1 + (1-\theta)\cdot n_2 = \theta\cdot\frac{(\pi_O - P_O)/\theta}{\pi_N + \pi_O - P_N - P_O} + (1-\theta)\cdot 0 = \frac{\pi_O - P_O}{\pi_N + \pi_O - P_N - P_O} \quad (9) .$$

One interesting property of the Bayesian equilibria of this game under incomplete information is that, for a given set of parameters, only one of them exists. Indeed, the situation is such that, if "$\theta < (\pi_O\text{-}P_O)/(\pi_N+\pi_O\text{-}P_N\text{-}P_O)$", then case A equilibrium exists and case B equilibrium does not, while if "$\theta > (\pi_O\text{-}P_O)/(\pi_N+\pi_O\text{-}P_N\text{-}P_O)$", then case B equilibrium exists and case A equilibrium does not. These relationships are the results of the following lemmas:

Lemma 2: *If the Bayesian equilibrium of the case A incomplete-information game exists, then it should hold that* "$\theta < (\pi_O\text{-}P_O)/(\pi_N+\pi_O\text{-}P_N\text{-}P_O)$".

Proof: Under the Bayesian equilibrium of case A, kicker 2 should play NS with a positive probability. Therefore it should hold that:

$$n_2 = \frac{(\pi_O - P_O) - (\pi_N - P_N)\cdot\theta/(1-\theta)}{2\cdot\pi_O - P_N - P_O} > 0 \quad .$$

But if this is so, then it should also hold that:

$$\pi_O - P_O > (\pi_N - P_N)\cdot\theta/(1-\theta) \quad \Rightarrow \quad (\pi_O - P_O)\cdot(1-\theta) > (\pi_N - P_N)\cdot\theta \quad ;$$

which implies that:

equal to a particular real number is always zero (as there are infinite real numbers between zero and one).

$$\pi_O - P_O > (\pi_N + \pi_O - P_N - P_O) \cdot \theta \qquad \Rightarrow \qquad \theta < \frac{\pi_O - P_O}{\pi_N + \pi_O - P_N - P_O} \qquad \text{q.e.d.}$$

Lemma 3: *If the Bayesian equilibrium of the case B incomplete-information game exists, then it should hold that "$\theta > (\pi_O\text{-}P_O)/(\pi_N\text{+}\pi_O\text{-}P_N\text{-}P_O)$".*

Proof: Under the Bayesian equilibrium of case B, kicker 1 should play NS with a probability that is smaller than one. Therefore it should hold that:

$$n_1 = \frac{(\pi_O - P_O)/\theta}{\pi_N + \pi_O - P_N - P_O} < 1 \qquad .$$

But if this is so, then it should also hold that:

$$\pi_O - P_O < (\pi_N + \pi_O - P_N - P_O) \cdot \theta \qquad \Rightarrow \qquad \theta > \frac{\pi_O - P_O}{\pi_N + \pi_O - P_N - P_O} \qquad \text{q.e.d.}$$

3. Additional results

The model described in the previous section implies that, when there are two types of kickers and only one type of goalkeeper, and there is incomplete information, then the Bayesian equilibrium of the corresponding incomplete-information game generally implies that one of the kicker's types will choose a pure strategy and the other type will choose a mixed strategy, while the goalkeeper will also choose a mixed strategy (which is the same strategy that he would choose if he were facing the kicker that is playing a mixed strategy). If we compare this Bayesian equilibrium with the Nash equilibria that would occurred if the same games were played under complete information, we would see that in this situation the goalkeeper is typically worse off and one of the kicker's types is typically better off.

In order to perform the comparisons outlined in the previous paragraph, we should compare the expected scoring probabilities under different situations. From those comparisons we will see that, given the parameters that we use in our model, kicker 1 obtains a higher expected payoff (i.e., a higher expected scoring probability) than kicker 2 under a complete-information Nash equilibrium. When we turn to the incomplete-information Bayesian equilibria analyzed, we see that the kicker who chooses a mixed strategy obtains the same expected payoff than under complete information, while the kicker who chooses a pure strategy is strictly better off. Under case B, moreover, kicker 2 is able to obtain the same expected payoff than kicker 1.

The expected scoring probability of a particular kicker is simply the average of the scoring probabilities implied by the strategy that he chooses, weighted by the probabilities that the goalkeeper "guesses" that strategy and by the probability that the goalkeeper "does not guess" that strategy. When a kicker is playing a mixed strategy, then the expected scoring probability of both NS and OS should be the same. When he is playing a pure strategy, conversely, his expected scoring probability is the one associated to the pure strategy that he chooses, that has to be larger than the expected scoring probability of the alternative strategy.

Under complete information, kicker 1's expected scoring probability is equal to:

$$SP_1(CI) = P_N \cdot \nu_1 + \pi_N \cdot (1-\nu_1) = \pi_O \cdot \nu_1 + P_O \cdot (1-\nu_1) = \frac{\pi_N \cdot \pi_O - P_N \cdot P_O}{\pi_N + \pi_O - P_N - P_O} \quad (10) \, ;$$

while kicker 2's expected scoring probability is equal to:

$$SP_2(CI) = P_N \cdot \nu_2 + \pi_O \cdot (1-\nu_2) = \pi_O \cdot \nu_2 + P_O \cdot (1-\nu_2) = \frac{\pi_O{}^2 - P_N \cdot P_O}{2 \cdot \pi_O - P_N - P_O} \quad (11) \, .$$

Under incomplete information, the expected scoring probabilities for the kickers depend on the case that holds. Under case A Bayesian equilibrium, kicker 2 obtains the same expected scoring probability that he gets under complete information, because "$\nu_A = \nu_2$" and therefore he is indifferent between choosing NS and OS. Kicker 1, conversely, is strictly better off by choosing NS, which now gives him the following expected scoring probability:

$$SP_1(IA) = P_N \cdot \nu_A + \pi_A \cdot (1-\nu_A) = \frac{\pi_O \cdot (\pi_N + P_N) - P_N \cdot (\pi_N + P_O)}{2 \cdot \pi_O - P_N - P_O} \quad (12) \, .$$

Conversely, under case B, kicker 1 obtains the same expected scoring probability that he gets under complete information, because "$\nu_B = \nu_1$" and therefore he is indifferent between choosing NS and OS. The one who is strictly better off is kicker 2, who is now choosing OS and obtaining the following expected scoring probability:

$$SP_2(IB) = \pi_O \cdot \nu_B + P_O \cdot (1-\nu_B) = \frac{\pi_O \cdot (\pi_N + P_O) - P_O \cdot (\pi_O + P_N)}{\pi_N + \pi_O - P_N - P_O} = \frac{\pi_N \cdot \pi_O - P_N \cdot P_O}{\pi_N + \pi_O - P_N - P_O} \quad (13) \, .$$

The idea that kicker 1 is better off than kicker 2 under complete information comes from the fact that, under the assumptions used in this paper, both kickers obtain the same expected payoff in three of the four cells of the scoring-probability matrix (see table 1) while kicker 1 gets a higher payoff in the remaining cell (since "$\pi_N > \pi_O$"). As the goalkeeper adjusts his strategy to this situation, however, the relationship between the expected scoring probabilities that these two types of kickers induce is not so obvious when one observes the equilibrium values gotten at equations 10 and 11. The proof that $SP_1(CI)$ is actually greater than $SP_2(CI)$, therefore, is given in the following proposition.

Proposition 1: *Under complete information, the expected scoring probability for kicker 1 is greater than the expected scoring probability for kicker 2.*

Proof: Under complete information, the expected scoring probability for kicker 1 is the same choosing NS and OS. Similarly, the expected scoring probability for kicker 2 is the same choosing NS and OS. Therefore we can write that:

$$SP_1(CI) = SP_1(CI/OS) = \pi_O \cdot \nu_1 + P_O \cdot (1-\nu_1) = P_O + (\pi_O - P_O) \cdot \nu_1 \quad ;$$

$$SP_2(CI) = SP_2(CI/OS) = \pi_O \cdot \nu_2 + P_O \cdot (1-\nu_2) = P_O + (\pi_O - P_O) \cdot \nu_2 \quad .$$

As we assume that "$\pi_O > P_O$", and we know from lemma 1 that "$\nu_1 > \nu_2$", then we also know that:

$$P_O + (\pi_O - P_O) \cdot \nu_1 > P_O + (\pi_O - P_O) \cdot \nu_2 \quad \Rightarrow \quad SP_1(CI) > SP_2(CI) \quad \text{q.e.d.}$$

A second comparison that we can make between expected scoring probabilities is the one that refers to $SP_1(CI)$ and $SP_1(IA)$, which is the theme of proposition 2. Finally, we can also prove that "$SP_2(IB) > SP_2(CI)$", and this is the theme of proposition 3.

Proposition 2: *Under case A Bayesian equilibrium with incomplete information, the expected scoring probability for kicker 1 is greater than the one that he obtains under complete information.*

Proof: Under complete information, the expected scoring probability for kicker 1 is the same choosing NS and OS. Conversely, the expected scoring probability for kicker 1 under case A with incomplete information is greater if he chooses NS, which is the pure strategy that he actually chooses in equilibrium. Therefore we can write that:

$$SP_1(CI) = SP_1(CI/NS) = P_N \cdot \nu_1 + \pi_N \cdot (1-\nu_1) = \pi_N - (\pi_N - P_N) \cdot \nu_1 \quad ;$$

$$SP_1(IA) = SP_1(IA/NS) = P_N \cdot \nu_A + \pi_N \cdot (1-\nu_A) = \pi_N - (\pi_N - P_N) \cdot \nu_A \quad .$$

By the definition of case A Bayesian equilibrium, we know that "$\nu_A = \nu_2$". As we also know that "$\pi_N > P_N$" (by assumption) and "$\nu_1 > \nu_2$" (from lemma 1), then is should hold that:

$$\pi_N - (\pi_N - P_N) \cdot \nu_1 < \pi_N - (\pi_N - P_N) \cdot \nu_A \quad \Rightarrow \quad SP_1(CI) < SP_1(IA) \quad \text{q.e.d.}$$

Proposition 3: *Under case B Bayesian equilibrium with incomplete information, the expected scoring probability for kicker 2 is greater than the one that he obtains under complete information.*

Proof: Under case B with incomplete information, the expected scoring probability for kicker 2 ($SP_2(IB)$) is the same than the expected scoring probability for kicker 1 under complete information ($SP_1(CI)$), since they are both equal to "$(\pi_N \cdot \pi_O - P_N \cdot P_O)/(\pi_N + \pi_O - P_N - P_O)$". As we know (from proposition 1) that "$SP_1(CI) > SP_2(CI)$", then this implies that "$SP_2(IB) > SP_2(CI)$", q.e.d.

4. Numerical example

The results that we have obtained in the two previous sections can be illustrated for a particular set of parameters. Using the estimates that appear in Coloma (2007), we will assume that "$\pi_N = 0.98$", "$\pi_O = 0.94$", "$P_N = 0.68$" and "$P_O = 0.48$". This implies that, under complete information, the equilibrium values for n_1, n_2, v_1 and v_2 are the following:

$$n_1 = \frac{0.94 - 0.48}{0.98 + 0.94 - 0.68 - 0.48} = 0.6053 \quad ; \qquad \nu_1 = \frac{0.98 - 0.48}{0.98 + 0.94 - 0.68 - 0.48} = 0.6579 \quad ;$$

$$n_2 = \nu_2 = \frac{0.94 - 0.48}{2 \cdot 0.94 - 0.68 - 0.48} = 0.6389 \;\; ;$$

which is therefore an example of the theoretical result that we obtained, which states that "$v_1 > v_2 = n_2 > n_1$". Besides, the corresponding expected scoring probabilities under this complete-information situation are the following:

$$SP_1(CI) = \frac{0.98 \cdot 0.94 - 0.68 \cdot 0.48}{0.98 + 0.94 - 0.68 - 0.48} = 0.7826 \quad ; \qquad SP_2(CI) = \frac{0.94^2 - 0.68 \cdot 0.48}{2 \cdot 0.94 - 0.68 - 0.48} = 0.7739 \quad .$$

If we now turn to the incomplete-information situation, we have two possible cases depending on the fact that θ is either greater than or smaller than 0.6053. When "$\theta < 0.6053$" (case A), it will hold that:

$$n_1 = 1 \quad ; \qquad \nu_A = \nu_2 = 0.6389 \quad ; \qquad n_2 = 0.6389 - \frac{0.4167 \cdot \theta}{1 - \theta} \quad ;$$

whereas, if "$\theta > 0.6053$" (case B), it will hold that:

$$n_2 = 0 \quad ; \qquad \nu_B = \nu_1 = 0.6579 \quad ; \qquad n_1 = \frac{0.6053}{\theta} \quad .$$

As we already know from the results obtained in section 3, "$SP_2(IA) = SP_2(CI) = 0.7739$" and "$SP_2(IB) = SP_1(IB) = SP_1(CI) = 0.7826$". By applying the formula that we have derived for equation 12, we can also find that:

$$SP_1(IA) = \frac{0.94 \cdot (0.98 + 0.68) - 0.68 \cdot (0.98 + 0.48)}{2 \cdot 0.94 - 0.68 - 0.48} = 0.7883 \qquad .$$

The incomplete-information case produces, as we have already seen, some results that depend on the value of θ, that is, on the proportion of kicker 1's that we have in the population under analysis. Figure 1 depicts the values of n_1, n_2 and v_M that we obtain as equilibrium values for all possible levels of θ, and in that figure we can see that n_1 tends to its complete-information level when θ tends to one, while n_2 tends to its complete-information level when θ tends to zero.

Figure 1: Equilibrium strategies under incomplete information

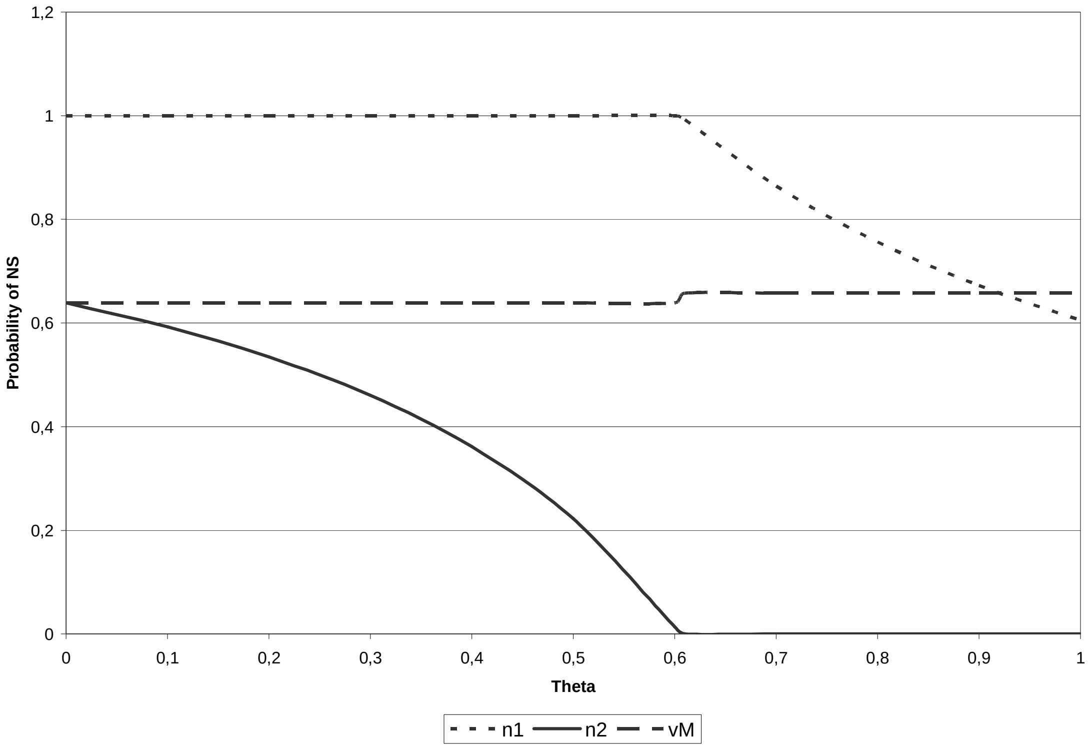
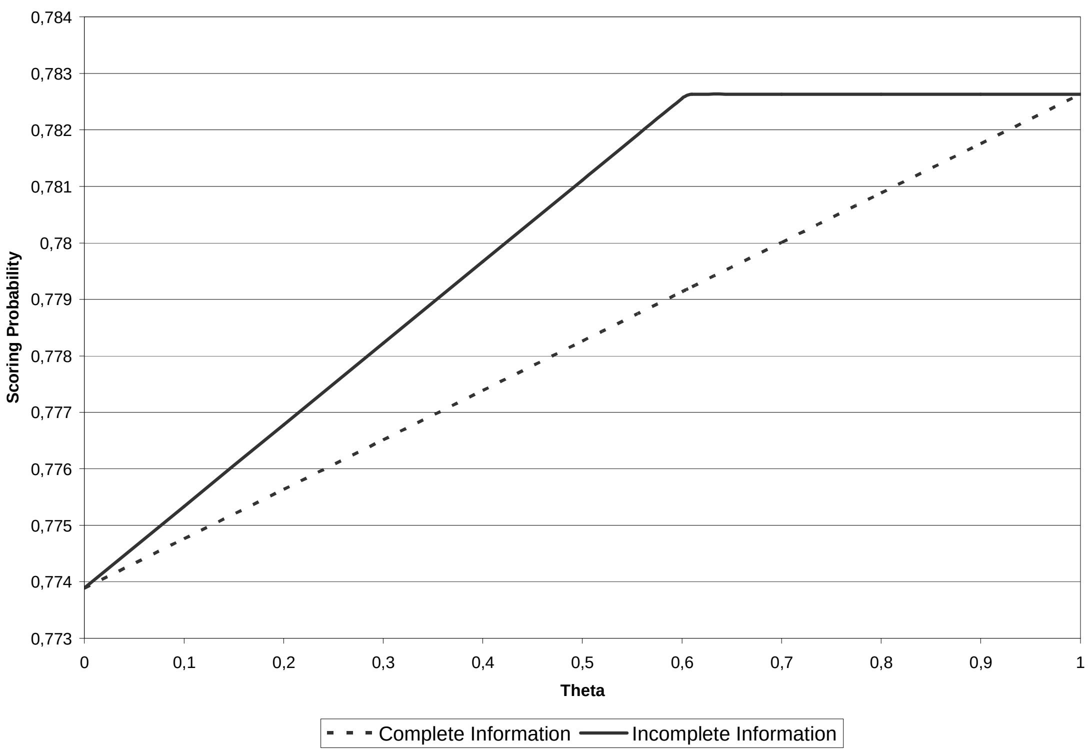


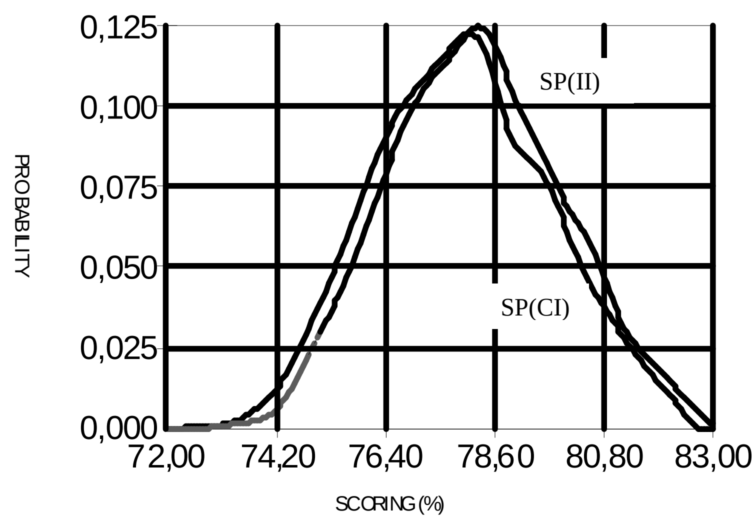
Figure 2: Average scoring probabilities



Correspondingly, figure 2 depicts the average scoring probability under complete and incomplete information for all possible levels of θ. In it we see that, unless "$\theta = 0$" or "$\theta = 1$", the average scoring probability is higher under incomplete information. We also see that, when θ increases, the average scoring probability under complete information also increases (since "$SP_1(CI) > SP_2(CI)$", and the average scoring probability is no other thing than "$\theta \cdot SP_1(CI) + (1-\theta) \cdot SP_2(CI)$"). The average scoring probability is also increasing in θ under incomplete information, but it reaches a maximum of 0.7826 when "$\theta = 0.6053$", and keeps that level for all values of θ that exceed that number.

To see how these values change under different scenarios, we also ran a Montecarlo simulation assuming a triangular distribution for the values of π_N, π_O, P_N and P_O, and a uniform distribution for the value of θ. After 10,000 iterations, we built a cruve with the distribution of the values for the average scoring probabilities under complete information (*SP(CI)*) and under incomplete information (*SP(II)*)[3]. Those distributions are represented on figure 3, which shows that, as expected, the distribution for *SP(II)* stochastically dominates the distribution for *SP(CI)*.

Figure 3: Simulation results for *SP(CI)* and *SP(II)*



[3] The four triangular distributions used are symmetric, and their respective intervals are [1, 0.96], [0.96, 0.92], [0.78, 0.58] and [0.58, 0.38]. The uniform distribution for θ is defined for the interval [0.3, 0.7]. The simulations were run using the @Risk 3.5 software.

5. Final remarks

The main conclusions of this paper have to do with the idea that, in some cases, the outcomes of a situation in which a soccer goalkeeper faces a kicker at a penalty kick can be explained as the result of an incomplete-information game. In those cases, the relevant solution concept is no longer the mixed-strategy Nash equilibrium of the game but the corresponding Bayesian equilibrium, since at least one of the players (e.g., the goalkeeper) is facing uncertainty about his opponent's type.

In the simplified model that we presented, we see that, under incomplete information, the typical situation is that at least one of the player's types (i.e., one of the kickers) chooses a pure strategy instead of a mixed strategy. This choice, which is almost impossible in equilibrium under complete information, arises because that type of kicker is actually responding to a strategy that the goalkeeper has designed for a different type of opponent. Being unable to distinguish among the different types, the goalkeeper has to play the same strategy against every opponent, and this is why some types of kickers may prefer a pure strategy. When we "mix" the strategies played by the different kickers, however, we end up with a sort of "average kicker strategy" which implies different probabilities for the different available pure strategies, and this average strategy has to be such that the goalkeeper is indifferent between playing the pure strategies that he mixes when he chooses his own best response to the "expected kicker".

The strategies implied by the Bayesian equilibrium of a game under incomplete information are typically not the same than the ones that the same players would choose if they were facing a complete-information game. Moreover, the relative lack of information that the goalkeeper faces in a situation of incomplete information makes that the average scoring probability is higher than under a situation of complete information. This is equivalent to say that, on average, the kicker is better off under incomplete information (i.e., under the kind of incomplete information that we have analyzed in this paper) and the goalkeeper is worse off.

The comparison between the Nash equilibria obtained under complete information and the Bayesian equilibrium obtained under incomplete information could also be useful to find the "value of information" in this game. As goalkeepers' payoffs are the complements of the scoring probabilities, then the value of knowing the true characteristics of a kicker can be measured as the difference between the expected scoring probability under complete and incomplete information. This difference, as we saw, is smaller if we are in a situation in which uncertainty is small (i.e., when θ is very close to zero or to one) and becomes larger when we approach the level of θ where the Bayesian equilibrium of the game changes from case A to case B. The difference will also be larger, of course, if the different types of kickers are "more different" among themselves.

Another virtue that we think that the incomplete-information approach could have is to solve some puzzles that the empirical literature on penalty kicks has discovered. Indeed, the Nash equilibrium concept has performed quite well to explain the data collected in several studies (e.g., Chiappori, Levitt and Groseclose, 2002; Palacios-Huerta, 2003; Baumann, Friehe and Wedow, 2011), but has failed in other cases. In a paper by Bar-Eli et al. (2007), for example, the authors have found, using data from several professional tournaments, that goalkeepers tend to stay in the center of the goal less often than what they would be supposed to do under the complete-information Nash equilibrium solution, and that they could therefore increase their expected payoff by choosing to stay in the center rather than jumping to the right or to the left[4]. This contradiction, we believe, could be solved if we assume that goalkeepers are actually facing two types of kickers (one of which always shoots center while the other one almost never does). If the proportion of "center-shooters" is relatively small (as it may be in Bar-Eli's sample, because the total fraction of kicks to the center is 0.287), then the optimal strategy for the goalkeeper might be to respond basically to the shots by the kickers who almost never choose the center, and this implies that the goalkeeper will almost never choose the center himself.

The main analytical problem of introducing incomplete information into the penalty-kick game, however, may paradoxically be its extreme capacity to match the data. Indeed, if we build a game of incomplete information that postulates different types of players and we arbitrarily use different probabilities for those types, then we could probably match any dataset on penalty kicks with a particular Bayesian equilibrium. If that is the case, then many of the empirical tests that the penalty-kick game-theoretic literature has designed will become useless, since it would actually be impossible to distinguish a Bayesian equilibrium from a situation in which the players are not choosing their strategies rationally.

[4] The explanation of Bar-Eli et al. for this choice is that goalkeepers are actually not minimizing expected scoring probability but following a behavior prescribed by the so-called "norm theory". That behavior implies that people tend to be less annoyed if they fail in a certain task after following the norms prescribed by society. In the penalty-kick case, Bar-Eli et al. consider that the social norm for the goalkeeper is to jump to one of the sides, whereas staying in the center of the goal could be seen as a kind of "inaction" which is not recommended by the social norms (although in this case it could imply a higher expected payoff).

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