12.1 Extending Black-Scholes

Consider the difference between a stock that provides a continuous dividend yield equal to \( q \) per annum and a similar stock that provides no dividends. When valuing a European option lasting for time \( T-t \) on a stock providing a known dividend yield equal to \( q \), we reduce the current stock price from \( S \) to \( Se^{-q(T-t)} \) and then value the option as though the stock pays no dividends.

\[
\begin{align*}
\text{Bounds for Option Prices} \\
\text{Consider the problem of determining bounds for the price of a European option on a stock providing a dividend yield equal to } q. \text{ Substituting } Se^{-q(T-t)} \text{ for } S, \text{ we see that a lower bound for the European call option price, } c, \text{ is given by}
\end{align*}
\]

\[
c > \max\left(Se^{-q(T-t)} - X e^{-r(T-t)}, 0\right)
\]

To obtain a lower bound for a European put option, we can similarly replace \( S \) by \( Se^{-q(T-t)} \) to get

\[
p > \max\left(X e^{-r(T-t)} - Se^{-q(T-t)}, 0\right)
\]

\[
\text{Put-Call Parity} \\
\text{Replacing } S \text{ by } Se^{-q(T-t)} \text{ we obtain put-call parity for an option on a stock providing a continuous dividend yield equal to } q:
\end{align*}
\]

\[
c + X e^{-r(T-t)} = p + Se^{-q(T-t)}
\]

12.2 Pricing Formulas

\[
c = Se^{-q(T-t)}N(d_1) - X e^{-r(T-t)}N(d_2) \quad (12.4)
\]

\[
p = X e^{-r(T-t)}N(-d_1) - Se^{-q(T-t)}N(-d_2) \quad (12.5)
\]

Since

\[
\ln\left(\frac{Se^{-q(T-t)}}{X}\right) = \ln\frac{S}{X} - q(T-t)
\]

\[
d_1 \text{ and } d_2 \text{ are given by}
\]

\[
d_1 = \frac{\ln(S/X) + (r - q + \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}
\]

\[
d_2 = \frac{\ln(S/X) + (r - q - \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}} = d_1 - \sigma\sqrt{T-t}
\]

If the dividend yield is not constant during the life of the option, equations (12.4) and (12.5) are still true, with \( q \) equal to the average annualized dividend yield during the life of the option.

Risk-Neutral

In a risk-neutral world, the total return from the stock must be \( r \). The dividends provide a return of \( q \). The expected proportional growth rate in the stock price must therefore be \( r - q \). To value a derivative dependent on a stock providing a continuous dividend yield equal to \( q \), we therefore set the expected growth rate of the stock equal to \( r-q \) and discount the expected payoff at rate \( r \).

12.3 Options on Stock Indices

Quotes

All are settled in cash rather than by delivering the securities underlying the index. This means that upon exercise of the option, the holder of a call option receives \( S - X \) in cash and the writer of the option pays this amount in cash, where \( S \) is the value of the index and \( X \) is the strike price. Similarly, the holder of a put option receives \( X - S \) in cash and the writer of the option pays this amount in cash. The cash payment is based on the index value at the end of the day on which the exercise instructions are issued.

Portfolio Insurance

Index options can be used by portfolio managers to limit their downside risk. Suppose that the value of an index is \( S \). Consider a manager in charge of a well-diversified portfolio which has a \( \beta \) of 1.0 so that its value mirrors the value of the index. If \( S \) is the value of the index, \( \beta \) puts contracts should be purchased for each 100S dollars in the portfolio. The strike price should be the value the index is expected to have when the value of the portfolio reaches the insured value.

Valuation

When options on stock indices are valued, it is usual to assume that the stock index follows geometric Brownian motion. This means that equation (12.4) and (12.5) can be used to value European call and put options on an index with \( S \) equal to the value of the index, \( \sigma \) equal to the volatility of the index, and \( q \) equal to the dividend yield on the index.
Equations (12.4) and (12.5) were presented on the assumption that dividends are paid continuously and that the rate at which they are paid is constant. In fact, both of these assumptions can be relaxed. All that is required is that we be able to estimate the dividend yield in advance.

As an alternative to estimating future dividend yields, we can attempt to predict the absolute amounts of the dividend that will be paid. The basic Black-Scholes formula can be used with $S$ set equal to the initial value of the stock index less the present value of the dividends. This is difficult to implement for a broadly based stock index since it requires a prediction of the dividends expected on every stock underlying the index.

12.4 Currency Options

Whereas a forward contract locks in the exchange rate for a future transaction, an options provides a type of insurance. Of course, insurance is not free. It costs nothing to enter into a forward transaction, while options require that a premium be paid up front.

Valuation

To value currency options, we define $S$ as the spot exchange rate. We assume that exchange rates follow the same type of stochastic process as a stock: geometric Brownian motion. We define $\sigma$ as the volatility of the exchange rate and $r^*$ as the risk-free rate of interest in the foreign country.

The European call price, $c$, and put price, $p$, are therefore given by

\begin{align*}
  c &= S e^{-r^* (T-t)} N(d_1) - X e^{-r (T-t)} N(d_2) \quad (12.6) \\
  p &= X e^{-r (T-t)} N(-d_2) - S e^{-r^* (T-t)} N(-d_1) \quad (12.7)
\end{align*}

Where

\begin{align*}
  d_1 &= \frac{\ln(S/X) + (r-r^*+\sigma^2/2)(T-t)}{\sigma \sqrt{T-t}} \\
  d_2 &= \frac{\ln(S/X) + (r-r^*-\sigma^2/2)(T-t)}{\sigma \sqrt{T-t}} = d_1 - \sigma \sqrt{T-t}
\end{align*}

Both the domestic interest rate, and the foreign interest rate, are assumed to be constant and the same for all maturities. The forward rate, $F$, for a maturity $T$ is given by

\[ F = S^{(r-r^*)(T-t)} \]

This enables equations (12.6) and (12.7) to be simplified to

\begin{align*}
  c &= e^{-r (T-t)} \left[ FN(d_1) - XN(d_2) \right] \quad (12.8) \\
  p &= e^{-r (T-t)} \left[ XN(-d_2) - FN(-d_1) \right] \quad (12.9)
\end{align*}

Where

\begin{align*}
  d_1 &= \frac{\ln(F/X) + (\sigma^2/2)(T-t)}{\sigma \sqrt{T-t}} \\
  d_2 &= \frac{\ln(F/X) - (\sigma^2/2)(T-t)}{\sigma \sqrt{T-t}} = d_1 - \sigma \sqrt{T-t}
\end{align*}

In some circumstance it is optimal to exercise American currency options prior to maturity. Thus American currency options are worth more than their European counterparts. In general, call options on high-interest currencies and put options on low-interest currencies are the most likely to be exercised prior to maturity. This is because a high-interest currency is expected to depreciate relative to the domestic currency and a low-interest currency is expected to appreciate relative to it.

12.5 Futures Options

Options on futures require the delivery of an underlying futures contract when exercised. If a call futures option is exercised, the holder acquires a long position in the underlying futures contract plus a cash amount equal to the current futures price minus the strike price. If a put futures option is exercised, the holder acquires a short position in the underlying futures contract plus a cash amount equal to the strike price minus the current futures price.

Reasons for the Popularity of Futures Options

Futures options are more attractive to investors than options on the underlying asset when it is cheaper or more convenient to deliver futures contract on the asset rather than the asset itself. An important point about a futures option is that the exercise of the option does not usually lead to delivery of the underlying asset, since in most circumstances the underlying futures contract is closed out prior to delivery.

Black’s Model

This model is based on the underlying assumption that the futures price, $F$, follows geometric Brownian motion:

\[ dF = \mu F dt + \sigma F dz \]

This assumption leads to the futures price being treated in the same way as a security providing a continuous dividend yield equal to $r$. The European call price, $c$, and European put price, $p$, for a futures option are therefore given by equations (12.4) and (12.5) with $S$ replaced by $F$ and $q = r$. 

\[ c = e^{-r(T-t)} \left[ FN(d_1) - XN(d_2) \right] \]  
\[ p = e^{-r(T-t)} \left[ XN(-d_2) - FN(-d_1) \right] \] 

Where

\[ d_1 = \frac{\ln(F/X) + (\sigma^2/2)(T-t)}{\sigma \sqrt{T-t}} \]
\[ d_2 = \frac{\ln(F/X) - (\sigma^2/2)(T-t)}{\sigma \sqrt{T-t}} = d_1 - \sigma \sqrt{T-t} \]

**The Expected Growth Rate of a Futures Price**

The expected growth rate in the price of a stock that pays dividends at rate \( q \) is \( r - q \) in a risk-neutral world. Since a futures price behaves like a stock where the dividends yield \( q \), equals \( r \), it follows that the expected growth rate in futures price in a risk-neutral world is zero. This is as might be expected. It cost nothing to enter into a futures contract. The expected gain to the holder of a futures contract in a risk-neutral world should therefore be zero.

Since the expected growth rate of the futures price is zero,

\[ F = E(F_T) \]

Since \( F_t = S_t \) it follows that

\[ F = E(S_t) \]

We have therefore shown that for all assets the futures price equals the expected future spot price in a risk-neutral world.

**Put-Call Parity**

A put-call parity relationship for European futures options can be derived in a similar way as for ordinary options. If \( F_T \) is the futures price at maturity, a European call plus an amount of cash equal to \( Xe^{-r(T-t)} \) has the terminal value

\[ \max(F_T - X, 0) + X = \max(F_T, X) \]

An amount of cash equal to \( Fe^{-r(T-t)} \) plus a futures contract plus a European put option has a terminal value

\[ F + (F_T - F) + \max(X - F_T, 0) = \max(X, F_T) \]

Since the two portfolios are equivalent at maturity, it follows that they are worth the same today. The futures contract is worth zero today. Hence

\[ c + Xe^{-r(T-t)} = p + Fe^{-r(T-t)} \]

**European Futures Options versus European Spot Options**

The futures price of any asset equals its spot price at maturity of the futures contract. It follows that a European futures option is worth the same as the corresponding European option on the underlying asset if the futures contract has the same maturity as the option.

**American Futures Options versus American Spot Options**

Traded futures options are in practice usually American. Assuming that the risk-free rate of interest is positive, there is always some chance that it will be optimal to exercise an American futures option early. American futures options are therefore worth more than their European counterparts. Unfortunately, no analytical formulas are available for valuing American futures options.

It is not generally true that an American futures option is worth the same as the corresponding American option on the underlying asset when the futures and options contract have the same maturity.