Crecimiento Economico
Class 1
Solow's model

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1 Economic growth: Class 1

.Crecimiento Economico

- Curso
  - que vamos a cubrir
  - que deben hacer ustedes
  - que libros vamos a usar
- Problema economica
  - Cuales son las causas de crecimiento economico
  - Porque algunos paises crecen mas rapido que otros
  - Porque hay paises ricos y pobres
  - Hay politicas que gobiernos pueden usar para que su pais crece mas rapido
    - en el corto o largo plazo
- Pensamos en terminos de modelos
- Primero: algunos datos (mas durante el proximo clase)
  - Porque el estudio del crecimiento importa para Argentina

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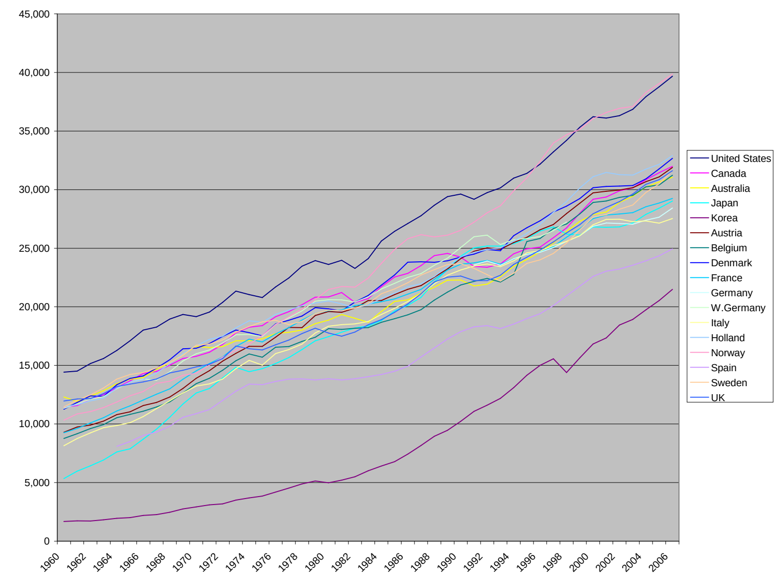
.Where does growth come from?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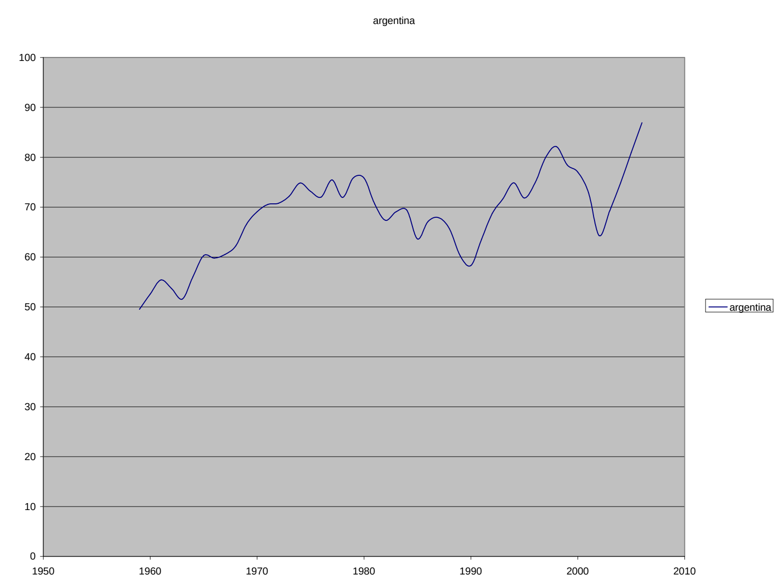
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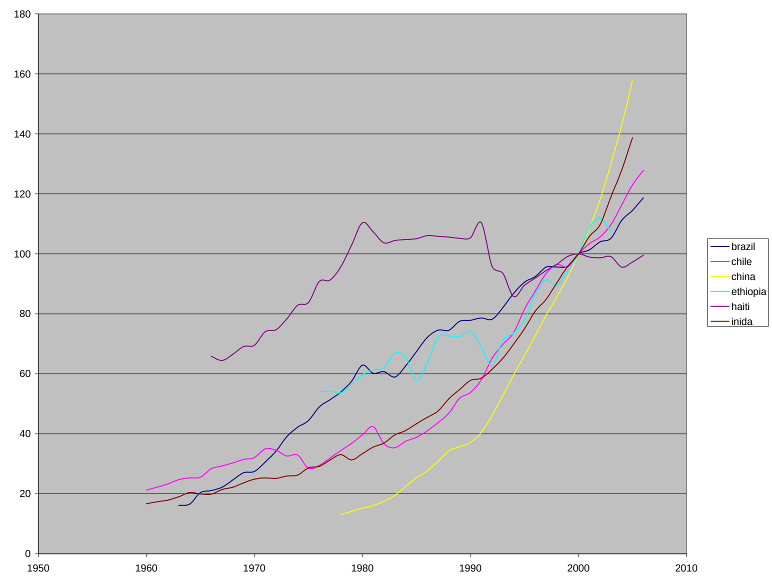
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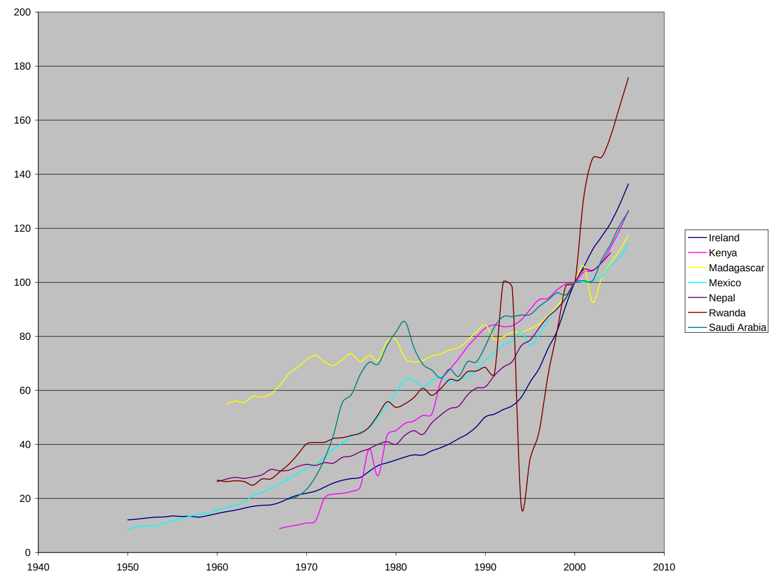
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- Economic view
- Production function (Cobb-Douglas)

$$Y_t = A_t K_t^{\theta} H_t^{1-\theta}$$

- complete derivative of the production function

$$dY_t = K_t^{\theta} H_t^{1-\theta} dA_t + \theta A_t K_t^{\theta-1} H_t^{1-\theta} dK_t + (1-\theta) A_t K_t^{\theta} H_t^{-\theta} dH_t$$

- Divide by Y_t

$$\frac{dY_t}{Y_t} = \frac{dA_t}{A_t} + \theta \frac{dK_t}{K_t} + (1-\theta) \frac{dH_t}{H_t}$$

- Output growth comes from growth in technology, capital, or labor
- Not shown: human capital: abilities of the workers (could be in technology)

Model: Robert Solow

Reference: Robert Solow: A Contribution to the Theory of Economic Growth (1956)

- Simplest model of economic growth
- microeconomic foundations (more or less)
- can give predictions about economic growth and its causes
- the concept of the Solow residual

Solow's growth model

Production function

$$Y_t = A_t F(K_t, N_t)$$

$Y_t =$ output, $K_t =$ stock of capital, $N_t =$ labor, $A_t = (1+\alpha)^t A_0 =$ the level of technology

Properties of $F(\cdot)$

1. twice continuously differentiable
2. homogenous of degree 1

$$\beta F(K_t, N_t) = F(\beta K_t, \beta N_t)$$

3. increasing in both arguments
4. Strictly concave: $F_i > 0$, $F_{ii} < 0$, $F_{ij} > 0$

5. Inada conditions: $F(K_t, 0) = 0$, $F(0, N_t) = 0$, $F_N(K_t, 0) = \infty$, $F_K(0, N_t) = \infty$, $F_N(K_t, \infty) = 0$, $F_K(\infty, N_t) = 0$

Solow model: population growth
Constant net growth rate of population (and labor) $= n$
The labor supply follows the rule:

$$N_{t+1} = (1+n) N_t$$

Solow model:
Capital accumulation process
Capital follows the accumulation rule

$$K_{t+1} = (1-\delta) K_t + I_t$$

I_t is the investment at time t
Savings rule

1. Savings is a fixed fraction of output (major simplification of Solow)

$$S_t = sY_t$$

   - No household optimization for savings decision
   - the parameter s is given exogenously

2. Equilibrium conditions for investment-savings

$$S_t = I_t$$

The full model

$$Y_t = A_t F(K_t, N_t)$$
$$K_{t+1} = (1-\delta) K_t + I_t$$
$$S_t = sY_t$$
$$S_t = I_t$$
$$N_{t+1} = (1+n) N_t$$

The first four can be combined to give the equation

$$K_{t+1} = (1-\delta) K_t + sA_t F(K_t, N_t)$$

Put everything into 'per unit of labor' terms

- Divide both sides of

$$K_{t+1} = (1-\delta) K_t + sA_t F(K_t, N_t)$$

  by $N_{t+1} = (1+n) N_t$ to get

$$\frac{K_{t+1}}{N_{t+1}} = \frac{(1-\delta) K_t}{(1+n) N_t} + \frac{sA_t F(K_t, N_t)}{(1+n) N_t}$$

- Defining per-worker terms: $k_t = K_t/N_t$, $y_t = Y_t/N_t$, this can be written as

$$k_{t+1} = \frac{(1-\delta)}{(1+n)}k_t + \frac{sA_t}{(1+n)}\frac{F(K_t, N_t)}{N_t}$$

.

- Because the production function is homogenous of degree 1,

$$\frac{F(K_t, N_t)}{N_t} = F\left(\frac{K_t}{N_t}, \frac{N_t}{N_t}\right) = F(k_t, 1) \equiv f(k_t),$$

- The Solow difference equation can be written as

$$k_{t+1} = G(k_t) = \frac{(1-\delta)}{(1+n)}k_t + \frac{sA_t}{(1+n)}f(k_t)$$

- Let $A_t = A_0$, a constant technology
- One can model with constant technological growth

$$A_t = (1+\alpha)^t A_0$$

(here $\alpha = 0$)

.

- Using the Cobb-Douglas production function and zero technological growth $f(k_t) = A_0 k_t^{\theta}$
- The Solow first difference equation is

$$k_{t+1} = \widehat{G}(k_t) = \frac{(1-\delta)}{(1+n)}k_t + \frac{sA_0}{(1+n)}k_t^{\theta}$$

- Given some initial k_0, the time path for the economy can be found
- Where the solution k to the equation

$$k = \widehat{G}(k) = \frac{(1-\delta)}{(1+n)}k + \frac{sA_0}{(1+n)}k^{\theta}$$

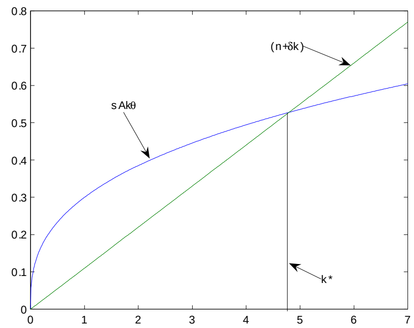
is a stationary state capital stock for the economy

Stationary points for economy

- When does

$$k = \frac{(1-\delta)}{(1+n)}k + \frac{sA_0}{(1+n)}k^{\theta}$$

- when $k = 0$, not a very interesting solution



- when

$$\left(\frac{(n+\delta)}{(1+n)}\right)k = \frac{sA_0}{(1+n)}k^{\theta}$$

$$k = \frac{sA_0}{(n+\delta)}k^{\theta}$$

$$k = \left(\frac{sA_0}{(n+\delta)}\right)^{\frac{1}{1-\theta}}$$

.Graphing the stationary points for economy

- we consider two way of doing this
- In the standard (undergraduate) way

$$k(n+\delta) = sA_0k^{\theta}$$

- Using the parameters $n = .01; \delta = .1; s = .3; \theta = .36, k^* = 4.7954$

.Graphing the stationary points for economy

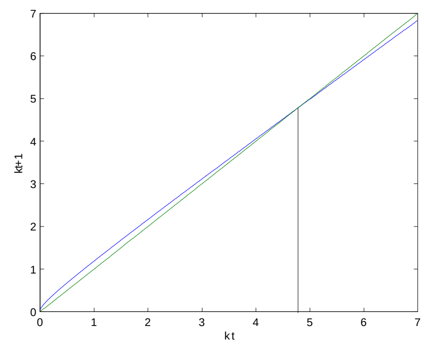
- Graph the difference equation

$$k_{t+1} = \frac{(1-\delta)}{(1+n)}k_t + \frac{sA_0}{(1+n)}k_t^{\theta}$$

- Graph in the 45 degree line (where $k_{t+1} = k_t$)

.Dynamics

- Reading the dynamics off of the difference equation graph
- for every k_t where $k_{t+1} > k_t$, the economy will grow



- for every k_t where $k_{t+1} < k_t$, the economy will shrink
- $k_t = 0$ is not a stable stationary state
  - a small positive shock will not return to $k_t = 0$
  - if some $k_t = \epsilon > 0$, then $k_{t+1} > k_t$
  - $k_t = 0$ is called a repeller
- $k_t = 4.7954$ is a stable stationary state
  - a small shock will return to $k_t = 4.7954$
  - $k_t = 4.7954$ is called an attracter
- Notice that these regions correspond to those on the other graph
  - where $k_{t+1} > k_t$, then $k\left(n+\delta\right) < sA_0k^\theta$
  - where $k_{t+1} < k_t$, then $k\left(n+\delta\right) > sA_0k^\theta$

Adding technological change: Balanced growth

- consider the model with growth of technology $= \alpha$, so

$$A_t = \left(1+\alpha\right)^t A_0$$

- First order equation is

$$\begin{aligned} K_{t+1} &= \left(1-\delta\right)K_t + sA_tK_t^\theta N_t^{1-\theta} \\ &= \left(1-\delta\right)K_t + s\left(1+\alpha\right)^t A_0K_t^\theta N_t^{1-\theta} \end{aligned}$$

- Convert to per capita (worker) by dividing both sides by $N_{t+1} = \left(1+n\right)N_t$

- get

$$\frac{K_{t+1}}{N_{t+1}} = \frac{(1-\delta)}{(1+n)}\frac{K_t}{N_t} + \frac{s\,(1+\alpha)^t A_0}{(1+n)}\frac{K_t^{\theta} N_t^{1-\theta}}{N_t}$$

or

$$k_{t+1} = \frac{(1-\delta)}{(1+n)}k_t + \frac{sA_0}{(1+n)}(1+\alpha)^t k_t^{\theta}$$

Balanced growth

- Balanced growth = constant growth rate (of per capita capital) = $\kappa = \kappa_t = \frac{k_{t+1}}{k_t}$
- Divide both sides of

$$k_{t+1} = \frac{(1-\delta)}{(1+n)}k_t + \frac{sA_0}{(1+n)}(1+\alpha)^t k_t^{\theta}$$

by k_t

- Get

$$\frac{k_{t+1}}{k_t} = \frac{(1-\delta)}{(1+n)} + \frac{sA_0}{(1+n)}(1+\alpha)^t k_t^{\theta-1}$$

- Want case where $\frac{k_{t+1}}{k_t}$ = a constant = κ
- Need to find value of κ and value of k_0

Balanced growth

- Set

$$\kappa = \frac{(1-\delta)}{(1+n)} + \frac{sA_0}{(1+n)}(1+\alpha)^t k_t^{\theta-1}$$

- with algebra get

$$\frac{\kappa - \frac{(1-\delta)}{(1+n)}}{\frac{sA_0}{(1+n)}} = (1+\alpha)^t k_t^{\theta-1}$$

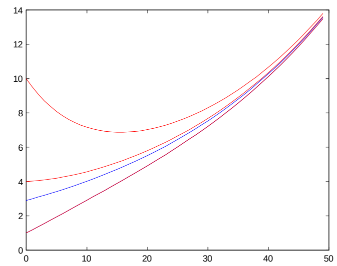
or

$$k_t^{1-\theta} = \frac{sA_0}{\kappa(1+n) - (1-\delta)}(1+\alpha)^t$$

- and finally

$$k_t = \left[\frac{sA_0}{\kappa(1+n) - (1-\delta)}\right]^{\frac{1}{1-\theta}}(1+\alpha)^{\frac{t}{1-\theta}}$$

Balanced growth



- To find κ, use the definition $\kappa = \kappa_t = \frac{k_{t+1}}{k_t}$ and get

$$\kappa = \frac{\left[\frac{sA_0}{\kappa(1+n)-(1-\delta)}\right]^{\frac{1}{1-\theta}} (1+\alpha)^{\frac{t+1}{1-\theta}}}{\left[\frac{sA_0}{\kappa(1+n)-(1-\delta)}\right]^{\frac{1}{1-\theta}} (1+\alpha)^{\frac{t}{1-\theta}}} = (1+\alpha)^{\frac{1}{1-\theta}}$$

- Put this into the equation for k_t and get

$$k_t = \left[\frac{sA_0}{(1+\alpha)^{\frac{1}{1-\theta}}(1+n)-(1-\delta)}\right]^{\frac{1}{1-\theta}} (1+\alpha)^{\frac{t}{1-\theta}}$$

.Graph of Balanced growth

- For an economy with $\delta = .1; s = .3; \theta = .36; \alpha = .02; n = .02;$
- k_0 is

$$k_0 = \left[\frac{.3 * 1}{(1+.02)^{\frac{1}{1-.36}}(1+.02)-(1-.1)}\right]^{\frac{1}{1-.36}} = 2.8916$$

- κ is

$$\kappa = (1+.02)^{\frac{1}{1-.36}} = 1.0314$$

.matlab program for balanced growth path

```
d=.1;
s=.3;
theta=.36;
a=.02;
n=.02;
k(1)=2.891 6;
for i=2:50
k(i)=(1-d)/(1+n)*k(i-1)+s*(1+a)^(i-1)*k(i-1)^theta/(1+n);
end
```

```
t=0:49;
plot(t,k)
```

.The golden rule saving rate

- We go back to the case where $\alpha = 0$, so no technology growth
- We want to know what s generates the highest long run (stationary state) consumption for the economy
- Consumption is the part of output that is not used for savings

$$c_t = (1 - s)\, y_t$$

- Higher s is not always best for consumption (if $s = 1$, consumption $= 0$)
- If $s = 0$, then $y = 0$ in stationary state (since there is on savings, k=0 in the stationary state and there is no output)

.The golden rule saving rate

- The standard way of solving it
- Given that

$$(n + \delta)\, k_t = sf(k_t)$$

as the usual condition for the stationary state and we want the point where

$$\begin{aligned} c_t &= y_t - sy_t \\ &= f(k_t) - (n + \delta)\, k_t \end{aligned}$$

is the maximum, this occurs when k_t^* is such that

$$f^{'}(k_t^*) = n + \delta.$$

- To find s^*, find the s where $(n + \delta)\, k_t^* = s^* f(k_t^*)$

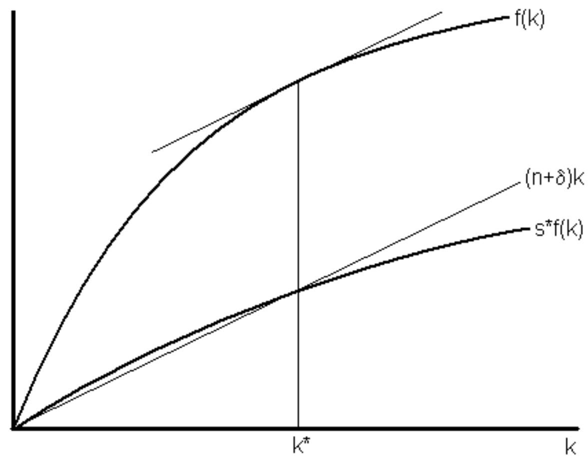
.The golden rule saving rate
.The golden rule saving rate

- An alternative way of finding the golden rule savings rate
- remember that stationary state capital is

$$k = \left(\frac{sA_0}{(n + \delta)}\right)^{\frac{1}{1-\theta}}$$

and stationary state output is

$$y = k^{\theta} = \left(\frac{sA_0}{(n + \delta)}\right)^{\frac{\theta}{1-\theta}}$$



- Economic problem: want to choose s to maximize $c = (1-s)\,y$, or

$$c = (1-s)\,y = (1-s)\left(\frac{sA_0}{(n+\delta)}\right)^{\frac{\theta}{1-\theta}}$$

.The golden rule saving rate

- First order conditions (the derivative of c with respect to s set equal to 0)

$$\begin{aligned}\frac{\partial c}{\partial s} &= -\left(\frac{sA_0}{(n+\delta)}\right)^{\frac{\theta}{1-\theta}} + \frac{\theta}{1-\theta}(1-s)\left(\frac{sA_0}{(n+\delta)}\right)^{\frac{\theta}{1-\theta}-1}\left(\frac{A_0}{(n+\delta)}\right) \\ &= 0\end{aligned}$$

- or

$$\left(\frac{sA_0}{(n+\delta)}\right)^{\frac{\theta}{1-\theta}} = \frac{\theta}{1-\theta}(1-s)\left(\frac{sA_0}{(n+\delta)}\right)^{\frac{\theta}{1-\theta}-1}\frac{1}{s}\left(\frac{sA_0}{(n+\delta)}\right)$$

- and

$$s = \frac{\frac{\theta}{1-\theta}}{\frac{1}{1-\theta}} = \theta$$

.The golden rule saving rate

- go back to the first method

- We have that
$$f^{'}(k_t^*) = n + \delta.$$
and
$$(n + \delta) k_t^* = s^* f(k_t^*)$$

- This implies that for the golden rule one needs to have that
$$f^{'}(k_t^*) k_t^* = s^* f(k_t^*)$$

- For the Cobb-Douglas production function, $A_0 k^{\theta}$,
$$f^{'}(k_t^*) = \theta A_0 k^{\theta - 1}$$
so
$$f^{'}(k_t) k_t = \theta A_0 k^{\theta-1} k_t = \theta f(k_t).$$

- In this case, $\theta = s$.

.The golden rule saving rate

- Why do we have $\theta = s$ with the Cobb-Douglas production function and does this carry over to other production functions?

- In general, it will not carry over. In the case of the Cobb-Douglas production function
$$f^{'}(k_t) k_t = \theta f(k_t)$$
and k_t^* disappears from the problem of
$$f^{'}(k_t^*) k_t^* = s^* f(k_t^*).$$
In general,
$$f^{'}(k_t) k_t \neq \theta f(k_t)$$
but rather something more complicated such as
$$f^{'}(k_t) k_t = g(k_t) f(k_t)$$
so that k_t^* is needed to get s^*. Since $n + \delta$ were needed to get k_t^*, then s^* will, in general, depend on n and δ.

.Concept of the Solow Residual

- Suppose that we have data for Y,K, and N over time for a country

- suppose that you calculate the average wage bill
  - wage bill $= wN = (1 - \theta) Y$ if production function is Cobb-Douglas
  - get θ

- Use the production function

$$Y_t = A_t F(K_t, N_t) = A_t K_t^{\theta} N_t^{1-\theta}$$

as

$$A_t = \frac{Y_t}{K_t^{\theta} N_t^{1-\theta}}$$

- The time path of A_t is the *Solow Residual*
  - it is the part of growth not explained by the growth of capital and labor
  - of course, this could be corrected for human capital (education)