1 Handling two step ahead expectations

Consider a model where x_t is a vector of state variables, y_t is a vector of jump variables and z_t is a vector of stochastic variables. What is different here is that two period ahead expecations of the state variables appear in the model. We can write the model in the form

$$\begin{aligned} Ax_t + Bx_{t-1} + Cy_t + Dz_t &= 0, \\ \widehat{F}x_{t+2} + Fx_{t+1} + Gx_t + Hx_{t-1} + \widehat{J}y_{t+2} + Jy_{t+1} + Ky_t + Lz_{t+1} + Mz_t &= 0, \end{aligned}$$

where the processes for the stochastic variables can be written as

$$z_t = Nz_{t-1} + \varepsilon_t,$$

$$z_{t+1} = Nz_t + \varepsilon_{t+1},$$

and

$$z_{t+2} = Nz_{t+1} + \varepsilon_{t+2}.$$

We are looking for policy funcitons of the form

$$\begin{aligned} x_t &= Px_{t-1} + Qz_t \\ y_t &= Rx_{t-1} + Sz_t. \end{aligned}$$

Substitute these policy functions into the model above (repeatedly substuting for the one and two step ahead forecasts and get the equation

$$\begin{aligned} 0 = \; & \widehat{F}PPPx_{t-1} + \widehat{F}PPQz_t + \widehat{F}PQNz_t + \widehat{F}\varepsilon_{t+1} + \widehat{F}QNNz_t + \widehat{F}QN\varepsilon_{t+1} + \widehat{F}Q\varepsilon_{t+2} \\ & + FPPx_{t-1} + FPQz_t + FQNz_t + FQ\varepsilon_{t+1} + GPx_{t-1} + GQz_t + Hx_{t-1} \\ & - \widehat{J}C^{-1}(AP+B)PPx_{t-1} - \widehat{J}C^{-1}(AP+B)PQz_t \\ & - \widehat{J}C^{-1}(AP+B)QNz_t - \widehat{J}C^{-1}(AP+B)Q\varepsilon_{t+1} \\ & - \widehat{J}C^{-1}(AQ+D)NNz_t - \widehat{J}C^{-1}(AQ+D)N\varepsilon_{t+1} - \widehat{J}C^{-1}(AQ+D)\varepsilon_{t+2} \\ & - JC^{-1}(AP+B)Px_{t-1} - JC^{-1}(AP+B)Qz_t \\ & - JC^{-1}(AQ+D)Nz_t - JC^{-1}(AQ+D)\varepsilon_{t+1} + KPx_{t-1} \\ & + KQz_t + LNz_t + L\varepsilon_{t+1} + Mz_t. \end{aligned}$$

from the expectations part and

$$[AP + B + CR]\, x_{t-1} + [AQ + CS + D]\, z_t = 0$$

from the part without expectations. This last can be solved as

$$\begin{aligned} R &= -C^{-1}(AP+B), \\ S &= -C^{-1}(AQ+D), \end{aligned}$$

but are in terms of the (yet unknown) P and Q matrices.

From the expectations equation, one can remove the expected values of the error terms greater than t, since they equal zero, and get

$$\begin{aligned}
0 \quad = \quad & \left[\left(\widehat{F}-\widehat{J}C^{-1}A\right)P^3+\left(F-\widehat{J}C^{-1}B-JC^{-1}A\right)P^2+\left[G-JC^{-1}B+K\right]P+H\right]x_{t-1} \\
& +\left[\widehat{F}-\widehat{J}C^{-1}A\right]QNNz_t \\
& +\left[\widehat{F}P+F-\widehat{J}C^{-1}AP-\widehat{J}C^{-1}B-JC^{-1}A\right]QNz_t \\
& +\left[\widehat{F}PP+FP+G-\widehat{J}C^{-1}APP-\widehat{J}C^{-1}BP-JC^{-1}AP-JC^{-1}B+K\right]Qz_t \\
& +\left[LN+M-\widehat{J}C^{-1}DNN-JC^{-1}DN\right]z_t
\end{aligned}$$

Now put together the terms in x_{t-1} and z_t. For a solution to hold for all values of x_{t-1} and z_t, we need

$$0=\left(\widehat{F}-\widehat{J}C^{-1}A\right)P^3+\left(F-\widehat{J}C^{-1}B-JC^{-1}A\right)P^2+\left[G-JC^{-1}B+K\right]P+H$$

and

$$\begin{aligned}
0 = & \left[\widehat{F}-\widehat{J}C^{-1}A\right]QNN+\left[\widehat{F}P+F-\widehat{J}C^{-1}AP-\widehat{J}C^{-1}B-JC^{-1}A\right]QN \\
& +\left[\widehat{F}PP+FP+G-\widehat{J}C^{-1}APP-\widehat{J}C^{-1}BP-JC^{-1}AP-JC^{-1}B+K\right]Q \\
& +\left[LN+M-\widehat{J}C^{-1}DNN-JC^{-1}DN\right]
\end{aligned}$$

Once P is known, the second equation can be solved using some characteristics of the $vec(\cdot)$ operator. Write the second equation as

$$\widetilde{A}QNN+\widetilde{B}QN+\widetilde{C}Q=\widetilde{D},$$

where the values of $\widetilde{A}$ to $\widetilde{D}$ are the parts in the appropriate brackets. Applying the $vec(\cdot)$ operator to both sides of this equation gives

$$vec\left(\widetilde{A}QNN\right)+vec\left(\widetilde{B}QN\right)+vec\left(\widetilde{C}Q\right)=vec\left(\widetilde{D}\right).$$

The following theorem is helpful.

Theorem 1 *Let A, B, C be matrices whose dimensions are such that the product ABC exists. Then*

$$vec(ABC)=(C'\otimes A)\cdot vec(B).$$

Applying the theorem and the equation becomes

$$\left(N'N'\otimes\widetilde{A}\right)vec\left(Q\right)+\left(N'\otimes\widetilde{B}\right)vec\left(Q\right)+\left(I\otimes\widetilde{C}\right)vec\left(Q\right)=vec\left(\widetilde{D}\right),$$

or

$$vec\left(Q\right)=\left[\left(N'N'\otimes\widetilde{A}\right)+\left(N'\otimes\widetilde{B}\right)+\left(I\otimes\widetilde{C}\right)\right]^{-1}vec\left(\widetilde{D}\right).$$

One only needs to put $vec(Q)$ into the appropriate matrix form to have the desired Q.

To find P, one needs to solve the cubic matrix polinomial. There are a number of methods for doing this. One, that follows Uhlig's method for quadratic matrix equations, follows.

A cubic polynomial is of the form

$$AP^3 - BP^2 - CP - D = 0,$$

where P is an $n \times n$ matrix such that $P = \Psi\Lambda\Psi^{-1}$, where Λ is a matrix with eigenvalues on the diagional and Ψ a matrix of the corresponding eigenvectors. Notice that with this notation, $P^2 = \Psi\Lambda\Psi^{-1}\Psi\Lambda\Psi^{-1} = \Psi\Lambda\Lambda\Psi^{-1} = \Psi\Lambda^2\Psi^{-1}$ and $P^3 = \Psi\Lambda\Psi^{-1}\Psi\Lambda\Psi^{-1}\Psi\Lambda\Psi^{-1} = \Psi\Lambda\Lambda\Lambda\Psi^{-1} = \Psi\Lambda^3\Psi^{-1}$. The matrices A, B, C, and D are all $n \times n$. Construct the $3n \times 3n$ matrices E and F as

$$E = \begin{bmatrix} B & C & D \\ I & 0 & 0 \\ 0 & I & 0 \end{bmatrix}$$

$$F = \begin{bmatrix} A & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix}$$

The solution to the generalized inverse problem is a set of 3n eigenvalues, λ, and the corresponding eigenvectors X and we write X as

$$X = \begin{bmatrix} X_{11} & X_{12} & X_{12} \\ X_{21} & X_{22} & X_{22} \\ X_{31} & X_{32} & X_{32} \end{bmatrix}.$$

The matrix Λ has the eigenvalues on the diagional and we can write that as

$$\Lambda = \begin{bmatrix} \Lambda_1 & 0 & 0 \\ 0 & \Lambda_2 & 0 \\ 0 & 0 & \Lambda_3 \end{bmatrix}.$$

The solution to the generalized eigenvalue problem can be written as

$$\begin{aligned} & \begin{bmatrix} B & C & D \\ I & 0 & 0 \\ 0 & I & 0 \end{bmatrix} \begin{bmatrix} X_{11} & X_{12} & X_{12} \\ X_{21} & X_{22} & X_{22} \\ X_{31} & X_{32} & X_{32} \end{bmatrix} \\ = & \begin{bmatrix} A & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} \begin{bmatrix} X_{11} & X_{12} & X_{12} \\ X_{21} & X_{22} & X_{22} \\ X_{31} & X_{32} & X_{32} \end{bmatrix} \begin{bmatrix} \Lambda_1 & 0 & 0 \\ 0 & \Lambda_2 & 0 \\ 0 & 0 & \Lambda_3 \end{bmatrix} \end{aligned}$$

which, written out, becomes

$$\begin{bmatrix} BX_{11}+CX_{21}+DX_{31} & BX_{12}+CX_{22}+DX_{32} & BX_{13}+CX_{23}+DX_{33} \\ X_{11} & X_{12} & X_{12} \\ X_{21} & X_{22} & X_{22} \end{bmatrix}$$

$$= \begin{bmatrix} AX_{11}\Lambda_1 & AX_{12}\Lambda_2 & AX_{13}\Lambda_3 \\ X_{21}\Lambda_1 & X_{22}\Lambda_2 & X_{22}\Lambda_3 \\ X_{31}\Lambda_1 & X_{32}\Lambda_2 & X_{32}\Lambda_3 \end{bmatrix}$$

The interesting parts of the equalities from the above matrix are

$$\begin{aligned} BX_{11}+CX_{21}+DX_{31} &= AX_{11}\Lambda_1 \\ X_{11} &= X_{21}\Lambda_1 \\ X_{21} &= X_{31}\Lambda_1. \end{aligned}$$

The last two give

$$\begin{aligned} X_{11} &= X_{31}\Lambda_1\Lambda_1 \\ X_{21} &= X_{31}\Lambda_1 \end{aligned}$$

so

$$BX_{31}\Lambda_1\Lambda_1 + CX_{31}\Lambda_1 + DX_{31} = AX_{31}\Lambda_1\Lambda_1\Lambda_1$$

and postmultiplying all this by X_{31}^{-1} gives

$$BX_{31}\Lambda_1\Lambda_1X_{31}^{-1} + CX_{31}\Lambda_1X_{31}^{-1} + DX_{31}X_{31}^{-1} = AX_{31}\Lambda_1\Lambda_1\Lambda_1X_{31}^{-1}.$$

Defining $P = X_{31}\Lambda_1X_{31}^{-1}$, one gets

$$BX_{31}\Lambda_1^2X_{31}^{-1} + CX_{31}\Lambda_1X_{31}^{-1} + D = AX_{31}\Lambda_1^3X_{31}^{-1}$$

or

$$BP^2 + CP + D = AP^3,$$

a P that solves the cubic polynomial equation.