The Taylor rule and issues of stability

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1 Introduction

The Taylor rule used in the book had two important characteristics: current inflation and output are the variables that the central bank used to determined the short term interest rate and the coefficients on the rule were representative of those recommended by Taylor, $a = .5$ and $b = .5$. With these coefficients and these variables, the Taylor rule produced smaller impulse response functions than did the fixed growth Friedman rule. A substantial literature has grown up around the issue of what should be the appropriate dating of the variables to be used and the coefficients that will generate stable solutions. In this literature, stability normally refers to uniqueness, that the model has a unique stable equilibrium under this central bank rule.

One frequently hears that central banks should base their Taylor rule on forecasts of inflation and output and that these forecasts should be made known to the public. This kind of policy is called inflation forecast targeting. The question of stability arises because there are doubts about these equilibria. Forward looking models can be fraught with problems related to bubbles, sunspots or self-confirming equilibria in cases where multiple equilibria are possible. One is therefore interested in those cases where a model with a Taylor rule has a unique or multiple (usually infinite in the case of linear models) equilibria. In particular, one wants to know the set of parameters of the Taylor rule for which there is only one equilibrium. In this case, the problems of bubbles do not occur.

We solve the log-linear rational expectations models using the techniques of Uhlig. The matrix quadratic equation whose stable root gives us the laws of motion for the state variables is solved using a generalized eigenvalue technique (see Section 8.8 of McCandless [1]). If there are exactly the same number of eigenvalues less than or equal to one as there are state variables, then the model has a unique stable solution. If there are more eigenvalues less than or equal to one, then there are more than one distinct solutions and all linear combinations of these solutions are also solutions of the model: implying that there are an infinite number of non-explosive solutions.

The literature on stability of the Taylor rule looks for the set of parameters on inflation and output that produces a unique solution. It does this by setting up the linear model and solving the generalized eigenvalue problem and counting the number of eigenvalues less than or equal to one and searching over a set of possible Taylor rule parameters on inflation and output for a specific model to see what set with unique solutions looks like.

Here we find the set of parameters for the Taylor rule that give a unique solution for the Taylor rule model given in Section 12.2 of McCandless [1] and some variations in which the date of inflation or output used in the rule is the expectations for period $t+1$ or historical from period $t-1$. The object is to compare the size of these sets and use that to determine what might make a sensible, stable Taylor rule for central bank practice. Here, for each model these sets are presented as a graph that separates the parameters that result in a unique equilibrium from those that result in multiple equilibria.

2 The model

Since the basic model is exactly the same as in Section 12.2, we begin here by rewriting the matrices of the model, moving the Taylor rule equation into the expectations part so that we can compare the basic model to one where the Taylor rule is based on rational expectations forecasts. The model where current values are used in the Taylor rule can be written as written with the set of state variables as $x_t = \left[\widetilde{K}_{t+1}, \widetilde{M}_t, \widetilde{P}_t\right]'$, the set of jump variables as $y_t = \left[\widetilde{r}_t, \widetilde{w}_t, \widetilde{Y}_t, \widetilde{C}_t, \widetilde{H}_t, \widetilde{N}_t, \widetilde{r}_t^n, \widetilde{r}_t^f, \widetilde{g}_t^M\right]'$, the stochastic variable as $z_t = \left[\widetilde{\lambda}_t, \widetilde{g}_t^f\right]$, and the system written as

$$\begin{aligned}
0 &= Ax_t + Bx_{t-1} + Cy_t + Dz_t, \\
0 &= E_t\left[Fx_{t+1} + Gx_t + Hx_{t-1} + Jy_{t+1} + Ky_t + Lz_{t+1} + Mz_t\right], \\
z_{t+1} &= Nz_t + \varepsilon_{t+1},
\end{aligned}$$

where

$$A = \begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & \overline{C} \\
\overline{K} & \overline{M/P} & \overline{r}^n\overline{N/P} - \overline{M/P} \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & -1 \\
0 & 0 & -\overline{w}\overline{H} \\
0 & 1 & 0
\end{bmatrix},$$

$$B = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -\frac{\overline{M/P}}{\overline{g}^M} & 0 \\ -(r+1-\delta)\overline{K} & 0 & 0 \\ -\theta & 0 & 0 \\ -(\theta-1) & 0 & 0 \\ -\theta & 0 & 0 \\ 0 & 0 & 0 \\ 0 & \overline{M/P}\left(1-\frac{1}{\overline{g}^M}\right) & 0 \\ 0 & -1 & 0 \end{bmatrix},$$

$$C = \begin{bmatrix} 0 & 1 & 0 & -1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & \overline{C} & 0 & \overline{N/P} & 0 & 0 & 0 \\ -\overline{r}\overline{K} & -\overline{w}\overline{H} & 0 & 0 & -\overline{w}\overline{H} & -\overline{r}^n\overline{N/P} & -\overline{r}^n\overline{N/P} & 0 & 0 \\ 0 & 1 & 0 & 0 & \theta & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & -(1-\theta) & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -(1-\theta) & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & -1 & 1 & 1 & -1 & 0 \\ 0 & -\overline{w}\overline{H} & 0 & 0 & -\overline{w}\overline{H} & \overline{N/P} & 0 & 0 & \overline{M/P} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix},$$

$$D = \begin{bmatrix} 0 & 0 \\ 0 & -\frac{\overline{M/P}}{\overline{g}^M} \\ 0 & 0 \\ -1 & 0 \\ -1 & 0 \\ -1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & -\frac{1}{\overline{g}^M} \end{bmatrix},$$

$$F = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$G = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & b\overline{g}^M \end{bmatrix},$$

$$H = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -b\overline{g}^M \end{bmatrix},$$

$$J = \begin{bmatrix} 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ \beta\overline{r} & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

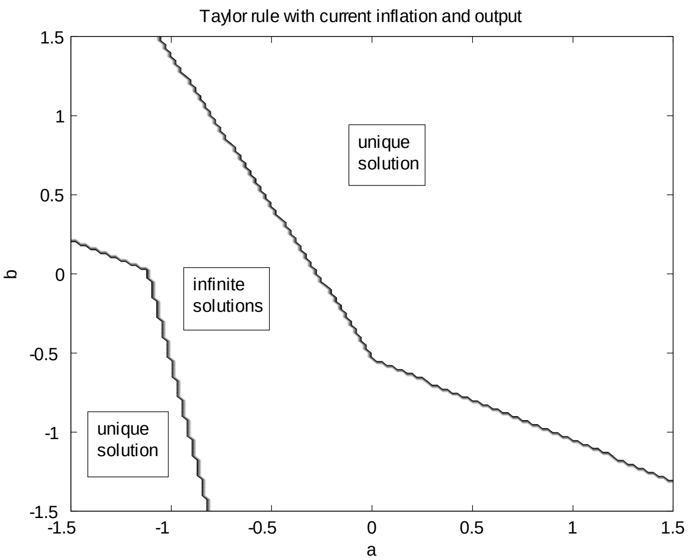

Figure 1: Purely current Taylor rule, $\overline{g}^M = 1.03$

$$K = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & a\overline{Y} & 0 & 0 & 0 & 0 & -\overline{r}^f & 0 \end{bmatrix},$$

$$L = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix},$$

$$M = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix},$$

and

$$N = \begin{bmatrix} \gamma & 0 \\ 0 & \pi^f \end{bmatrix}.$$

In this version of the model, the money growth rule is in the A, B, C, D matrices and the Taylor rule is in the matrices from F to M.

We consider values of a and b from the set $a, b \in [-1.5, 1.5]$. The unique solution parameter space for a Taylor rule of the form

$$r_t^f = a\left(Y_t - \overline{Y}\right) + b\left(\pi_t - \overline{\pi}\right) + \overline{r}^f$$

is indicated in Figure 1. There are two separate regions where the solutions are stable but these are very different solutions and the properties of the impulse response functions are quite different. That happens is that as a and b become smaller, one of the eigenvalues that is greater than one shrinks and the eigenvalue that strictly between one and zero grows until we reach a region where both are less than one. As a and b continue to become smaller, the eigenvalue that was initially stable becomes unstable and the other, now less than one, eigenvalue is chosen for the unique equilibrium. The Taylor rule implied by the values of a and b in the lower left hand corner of Figure 1 is quite odd, increases in output and inflation generate sharp reductions in the short term interest rate that the firms pay.

To find the set for the model where the Taylor rule is based on rational expectations forecasts of inflation and output, one modifies matrices F through K to be

$$F = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & b\overline{g}^M \end{bmatrix},$$

$$G = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & -b\overline{g}^M \end{bmatrix},$$

$$H = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$J = \begin{bmatrix} 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ \beta\overline{r} & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & a\overline{Y} & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$K = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\overline{r}^f & 0 \end{bmatrix}.$$

The only changes are in the last row of these matrices. Compare these matrices to the ones given above. The space of unique solutions for the version of the Taylor rule that uses rational expectations forecasts for both inflation and output, where the Taylor rule is

$$r_t^f = a\left(E_t Y_{t+1} - \overline{Y}\right) + b\left(E_t \pi_{t+1} - \overline{\pi}\right) + \overline{r}^f,$$

is given in Figure 2.

The set of unique solutions is substantially smaller than that with current variables, so much so that the coefficients used in Section 12.2 of McCandless [1], $a = .5$ and $b = .5$ is not in the set of unique solutions.

For a mixed Taylor rule, with current output and expected inflation, the rule can be written as

$$r_t^f = a\left(Y_t - \overline{Y}\right) + b\left(E_t \pi_{t+1} - \overline{\pi}\right) + \overline{r}^f,$$

and the set with unique solutions is given in Figure 3.

Using current output in the Taylor rule substantially expands the set of parameters that result in a unique stable solution.

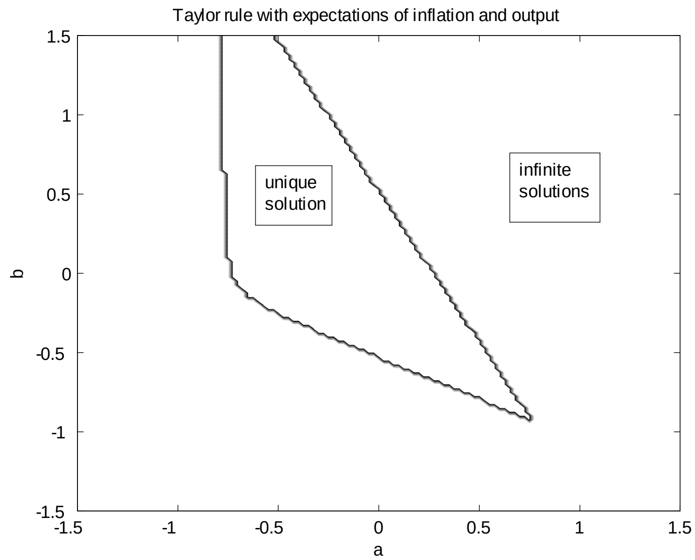

Figure 2: Purely expectational Taylor rule, $\overline{g}^M = 1.03$

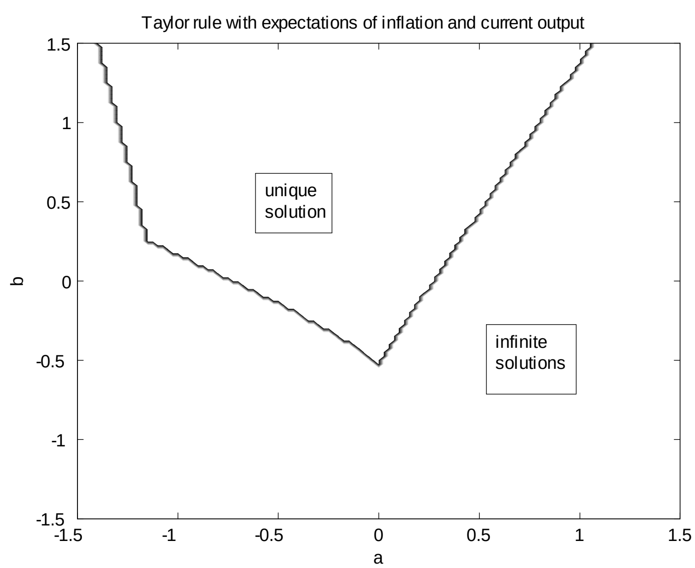

Figure 3: Taylor rule with expected inflation and current output, $\overline{g}^M = 1.03$

2.1 Lagged Taylor rule

A Taylor rule that uses lagged data requires a change in the set of state variables so that the time $t-2$ prices and time $t-1$ output are available to calculate past inflation and output. To write the model with a lagged Taylor rule implies increasing the state space to include lagged output and two period lagged prices. For a model with lagged inflation and output in the Taylor rule the log-linear version of the model is

$$\begin{aligned}
0 &= \widetilde{w}_t + \widetilde{P}_t - E_t\widetilde{P}_{t+1} - E_t\widetilde{C}_{t+1}, \\
0 &= \widetilde{w}_t - E_t\widetilde{w}_{t+1} + \beta\overline{r}E_t\widetilde{r}_{t+1}, \\
0 &= \widetilde{r}_t^n - \widetilde{w}_t + \widetilde{C}_t, \\
0 &= \overline{C}\widetilde{C}_t - \frac{\overline{M/P}}{\overline{g}^M}\widetilde{g}_t^f - \frac{\overline{M/P}}{\overline{g}^M}\widetilde{M}_{t-1} + \overline{N/P}\widetilde{N}_t + \overline{C}\widetilde{P}_t,
\end{aligned}$$

$$\begin{aligned}
0 &= \overline{M/P}\widetilde{M}_t + \left[\overline{r}^n\overline{N/P} - \overline{M/P}\right]\widetilde{P}_t + \overline{K}\widetilde{K}_{t+1} - \overline{w}\overline{H}(\widetilde{w}_t + \widetilde{H}_t) \\
&\quad -\overline{r}\overline{K}\widetilde{r}_t - (\overline{r}+1-\delta)\overline{K}\widetilde{K}_t - \overline{r}^n\overline{N/P}\widetilde{N}_t - \overline{r}^n\overline{N/P}\widetilde{r}_t^n, \\
0 &= \widetilde{w}_t + \widetilde{r}_t^f - \widetilde{\lambda}_t - \theta\widetilde{K}_t + \theta\widetilde{H}_t,
\end{aligned}$$

$$\begin{aligned}
0 &= \widetilde{r}_t - \widetilde{\lambda}_t - (\theta-1)\widetilde{K}_t - (1-\theta)\widetilde{H}_t, \\
0 &= \widetilde{Y}_t - \widetilde{\lambda}_t - \theta\widetilde{K}_t - (1-\theta)\widetilde{H}_t, \\
0 &= \widetilde{r}_t^n + \widetilde{N}_t - \widetilde{P}_t - \widetilde{r}_t^f - \widetilde{w}_t - \widetilde{H}_t,
\end{aligned}$$

$$\begin{aligned}
0 &= \overline{N/P}\widetilde{N}_t + \overline{M/P}\left(1 - \frac{1}{\overline{g}^M}\right)\widetilde{M}_{t-1} - \overline{w}\overline{H}\widetilde{P}_t \\
&\quad +\overline{M/P}\widetilde{g}_t^M - \overline{w}\overline{H}\widetilde{w}_t - \overline{w}\overline{H}\widetilde{H}_t \\
0 &= \widetilde{M}_t - \frac{1}{\overline{g}^M}\widetilde{g}_t^f - \widetilde{g}_t^M - \widetilde{M}_{t-1} \\
0 &= a\overline{Y}\widetilde{Y}_{t-1} + b\overline{g}^M\widetilde{P}_{t-1} - b\overline{g}^M\widetilde{P}_{t-2} - \overline{r}^f\widetilde{r}_t^f.
\end{aligned}$$

Notice that $\widetilde{P}_{t-2}$ and $\widetilde{Y}_{t-1}$ occur in the log-linear Taylor rule the last equation. Defining the set of state variables as $x_t = \left[\widetilde{K}_{t+1}, \widetilde{M}_t, \widetilde{P}_t, \widetilde{P}_{t-1}, \widetilde{Y}_t\right]'$, the set of jump variables as $y_t = \left[\widetilde{r}_t, \widetilde{w}_t, \widetilde{C}_t, \widetilde{H}_t, \widetilde{N}_t, \widetilde{r}_t^n, \widetilde{r}_t^f, \widetilde{g}_t^M\right]'$, and the stochastic variable as $z_t = \left[\widetilde{\lambda}_t, \widetilde{g}_t^f\right]$, the system can be written as

$$\begin{aligned}
0 &= Ax_t + Bx_{t-1} + Cy_t + Dz_t, \\
0 &= E_t\left[Fx_{t+1} + Gx_t + Hx_{t-1} + Jy_{t+1} + Ky_t + Lz_{t+1} + Mz_t\right], \\
z_{t+1} &= Nz_t + \varepsilon_{t+1},
\end{aligned}$$

where

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \overline{C} & 0 & 0 \\ \overline{K} & \overline{M/P} & \overline{r}^n \overline{N/P} - \overline{M/P} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & -\overline{w}\overline{H} & 0 & 0 \end{bmatrix},$$

$$B = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & -\frac{\overline{M/P}}{\overline{g}^M} & 0 & 0 & 0 \\ -(r+1-\delta)\overline{K} & 0 & 0 & 0 & 0 \\ -\theta & 0 & 0 & 0 & 0 \\ -(\theta-1) & 0 & 0 & 0 & 0 \\ -\theta & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & \overline{M/P}\left(1 - \frac{1}{\overline{g}^M}\right) & 0 & 0 & 0 \end{bmatrix},$$

$$C = \begin{bmatrix} 0 & 1 & -1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & \overline{C} & 0 & \overline{N/P} & 0 & 0 & 0 \\ -\overline{r}\overline{K} & -\overline{w}\overline{H} & 0 & -\overline{w}\overline{H} & -\overline{r}^n \overline{N/P} & -\overline{r}^n \overline{N/P} & 0 & 0 \\ 0 & 1 & 0 & \theta & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & -(1-\theta) & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -(1-\theta) & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & -1 & 1 & 1 & -1 & 0 \\ 0 & -\overline{w}\overline{H} & 0 & -\overline{w}\overline{H} & \overline{N/P} & 0 & 0 & \overline{M/P} \end{bmatrix},$$

$$D = \begin{bmatrix} 0 & 0 \\ 0 & -\frac{\overline{M/P}}{\overline{g}^M} \\ 0 & 0 \\ -1 & 0 \\ -1 & 0 \\ -1 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix},$$

$$F = \begin{bmatrix} 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$G = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix},$$

$$H = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & b\overline{g}^M & -b\overline{g}^M & a\overline{Y} \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \end{bmatrix},$$

$$J = \begin{bmatrix} 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ \beta\overline{r} & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$K = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\overline{r}^f & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix},$$

$$L = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix},$$

$$M = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & -\frac{1}{\overline{g}^M} \end{bmatrix},$$

and

$$N = \begin{bmatrix} \gamma & 0 \\ 0 & \pi^f \end{bmatrix}.$$

Notice that the parameters of the Taylor rule are in the third line of matrix H where they fall on time $t-1$ and $t-2$ prices and on time $t-1$ output.

The unique solution parameter space for the model with a lagged version of the Taylor rule is given in Figure 4.

The space of unique solutions is very large, including almost all the set over which we searched.

The results shown in Figure 4 are usually take to mean that a Taylor rule should include a lagged inflation component to "insure" that the model (and

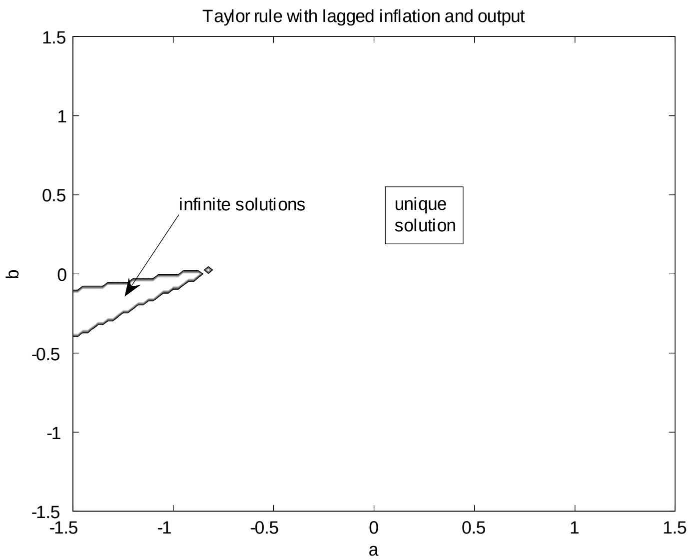

Figure 4: Purely lagged Taylor rule, $\overline{g}^M = 1.03$

possibly, the economy) has a unique solution. Figure 5 shows the space of unique solutions for a mixed Taylor rule of the form

$$r_t^f = a\left(Y_t - \overline{Y}\right) + .5b\left(E_t\pi_{t+1} - \overline{\pi}\right) + .5b\left(\pi_{t-1} - \overline{\pi}\right) + \overline{r}^f.$$

This is probably the Taylor most commonly used. Compare the set of unique solutions to that with all forward looking given in Figure 2 or with inflation expectations and current output as in Figure 3.

3 Reprise

The timing of the variables used in a Taylor rule has a serious impact on the space of parameter values that provide for a unique solution to the model. A Taylor rule with all lagged values has the largest space and one with pure rational expectations forecasts has the smallest. The version of the Taylor rule that is most commonly used in practice contains a linear combination of a forecast for inflation and lagged inflation, usually with lagged output. For the basic model of monetary policy with a financial sector, the commonly used model has a space of unique solutions almost as large as the purely lagged one.

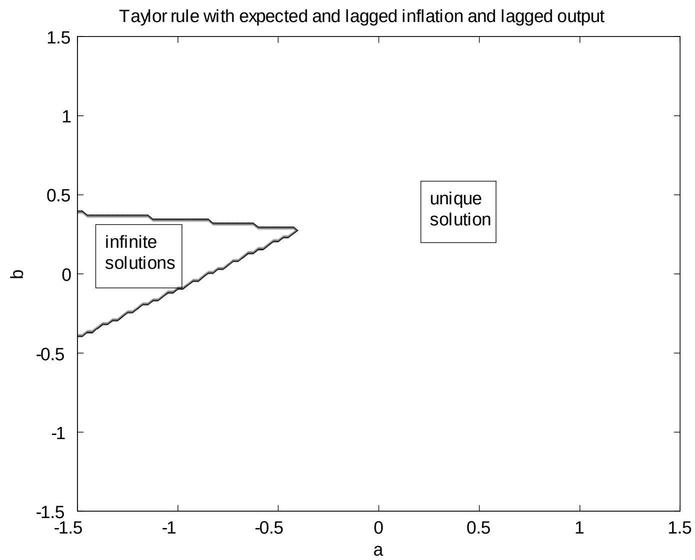

Figure 5: Set of unique solutions for Taylor rule with expected and lagged inflation, $\overline{g}^M = 1.03$

References

[1] McCandless, George (2008) *The ABCs of RBCs: An Introduction to Dynamic Macroeconomic Models*, Harvard University Press, Cambridge.