Chapter 1

Introducing Uncertainty

1.1 Introduction

There are several fairly simple ways of extending our model to allow for uncertainty. The development of our model with uncertainty follows the same form as the development of our earlier models. The main difference is not so much in terms of concept, but in terms of algebreic complexity. At the minimum, we add one additional dimension to the model by allowing at least two outcomes for environmental variables during old age. The dimensionality of the choice problem goes up accordingly.

Up to now, the future values of all of the environmental variables have been known to the members of each generation. This has been a result of our assumption of perfect foresight. Adding uncertainty means making the values for some of these future variables unknown and unknowable, at least to some degree.

Much of the early development of the theory of uncertainty involved gambling. Games of chance often have particularly simple structures. Flipping a coin and attempting to predict which side will face upwards has only two possible outcomes: heads or tails. Each of the plays of this game are independent, which means that they do not depend on the outcome of earlier plays of the game. If I flip a coin, whether it comes up heads or tails does not depend on whether it came up heads or tails on any of the earlier flips. The game can be repeated many times and the frequency with which heads or tails appears for any particular coin can be discovered. The limit of the frequency that heads appears (for example), as a percent of the total number of attempts, is called the probability of that event. With repeatable games of chance, the probabilities represent the outcomes of many repetitions, and can be thought of as representing some objective observation of the world.

In economics, Frank Knight made a distinction between risk and uncertainty. Risk was the term he applied to situations where the probabilities of events could be calculated and represented some objective observations on the world. Calculations of the cost of insurance against death at any particular age or against automobile accidents can be made based on observations of the population. Individuals can make agreements with firms or other individuals to protect themselves against these calculable risks.

Knight used the term, uncertainty, to denote events for which the probabilities can not be calculated objectively. Events that occur infrequently or have not yet ever occurred go into this catagory. Calculations of the return on investment in the production of a new commodity, the dangers associated with the meltdown of a nuclear power plant, and the policies of a new government to an inflation of 200% a month are dramatic examples of situations where objective probabilities are difficult to define because little prior experience is available. In these cases, any probability estimates should be thought of as being subjective.

Some economists have argued that even in cases where objective (large sample) probabilities exist, they might not be very relevant to individual behavior. For example, since individuals die only once, they are more interested in their particular position in the distribution of life expectancy than in the entire characterization of that distribution. However, individuals do respond to information about changes in behavior that change the distribution of the age at which one dies. The reduction over the last few years in the number of people who smoke cigarettes can be attributed to the information about the effects of cigarette smoking on life expectancy.

Modern formulations of the theory of decisions under uncertainty have managed to avoid most the the risk - uncertainty issues. We can speak of modelling behavior with individuals who maximize their expected utility based on subjective probabilities. To do this carefully, we need to define the environment somewhat differently than in the perfect foresight world.

1.2 The environment under uncertainty

We need to introduce three new elements into our model. These three elements are 1) states of nature, 2) actions, and 3) consequences. These three elements are not independent. An action will generate consequences in states of nature.

Definition 1 *The states of nature at time t are the set of all combinations of values for the environmental variables that can occur at time period t.*

At any time $t-1$, there are as many states of nature for time t as there are combinations of different outcomes that could occur for the environmental variables. Suppose that there are two states of nature possible at time t, one state in which the crop, $d(t)$, equals 1/3 and another state in which $d(t)$ equals 2/3. Suppose that all of the other environmental variables can have only one value at time t. Then there are only two states of nature. State "one" in which $d(t) = 1/3$ and all the other environmental variables have their time t value and state "two" in which $d(t) = 2/3$ and the other variables have the same value as in state one.

If environmental variables other than the crop size can have multiple values, then one will have more states of nature. Suppose that at time t, there are two different quantites of endowment possible for members of generation $t-1$ and at time $t-1$ it is not known which of these two possible endowments will occur. As an example, let these possible endowments be

$$\omega_{t-1}^h(t) = 2$$

or

$$\omega_{t-1}^h(t) = 3.$$

In that case there can be four possible states of nature for time t. These four states of nature (arbitrarily numbered) are

state 1: $d(t) = 1/3$ and $\omega_{t-1}^h(t) = 2$,
state 2: $d(t) = 1/3$ and $\omega_{t-1}^h(t) = 3$,
state 3:$d(t) = 2/3$ and $\omega_{t-1}^h(t) = 2$, and
state 4: $d(t) = 2/3$ and $\omega_{t-1}^h(t) = 3$.

In normal life, one can think in terms of the possible states of nature. In summer in Chicago, on any given day, it might be hot or cool, it might rain or not, and the Cubbies, when playing at home, might win or lose. Since baseball is not played in the rain, there are only six possible states of nature in this example. These states are

state 1: it is hot and doesn't rain and the Cubs win,
state 2: it is hot and doesn't rain and the Cubs lose,
state 3: it is cool and doesn't rain and the Cubs win,
state 4: it is cool and doesn't rain and the Cubs lose,
state 5: it is hot and rains, and
state 6: it is cool and rains.

Since the Cubs don't play in the rain, we can not define states of nature involving the results of a baseball game on a day when it rains. In the world of everyday life, there can be many (possibly infinitely many) possible states of nature. In one sense, when we have uncertainty, individuals need to know more than they needed to know when there was perfect foresight. They need to know all of the possible states of nature that can exist.

Definition 2 *The set of actions at time t is the full set of decisions that a member of generation t can make when young.*

The set of actions can be finite or infinite. In a very simple world, the set of actions might be to carry an umbrella or not to carry an umbrella. (The states of nature for which this might be an interesting set of actions is the set of states made up of "it will rain today" and "it will not rain today".) For our usual model with land and the two possible crop sizes, the set of actions for an individual is comprised of all of the physically possible quantities of land purchases.

Definition 3 *A consequence is the result at time $t+1$ of one action at time t in one state of nature at time $t+1$.*

Each action generates a consequence in each state of nature. The total number of consequences (there can be duplicates) are equal to the number of states of nature times the number of actions. Under this definition, each action generates a set of mutually exclusive consequences. Only one state of nature occurs at time $t+1$ and only one consequence will occur as a result of a specific action. At time t, it is not known which state of nature will occur at time $t+1$, so it is not known which consequence of any given action will occur.

The relationship between actions, states of nature, and consequences are shown in Figure X.1. Let a stand for actions $i = 1, 2$, and 3. Let s stand for states of nature $j = 1$ and 2. Let c^{ij} stand for the consequence of action i when state of nature j occurs.

Actions	s^1	s^2
a^1	c^{11}	c^{12}
a^2	c^{21}	c^{22}
a^3	c^{31}	c^{32}

Figure X.1

Exercise 1.2.1 *Describe the states of nature, actions, and consequences for each of the following situations: a. planning a picnic b. launching a rocket c. preparing for a date.*

1.3 Pure state contingent securities

Up to now, loans between individuals have been completely riskless. If one member of generation t lent some endowment to another member of generation t, the loan would be paid back the next period with the agreed upon interest. We did not allow any possibility of default or partial repayment.

Imagine the situation of a farmer. The farmer borrows from a bank (or someone not in the farming business) to plant a crop and the outcome (harvest size) will depend on which state of nature (mainly weather) that occurs at that farm. The harvest size for a particular farmer is not the only variable that determines the ability of that farmer to pay back the loan. The price the farmer will get for the crop depends on the state of nature that occurs over all of the farms that have planted that crop. There can be locally bad weather for a particular crop when nationally, the weather is quite good for that crop. In that case there is a large general supply of the good and a low price, while some farmers can have a small harvest. Likewise, some farmers may have a very good harvest in years when the national crop is quite small. In that year, these farmers will do particularly well.

We can make the gross interest rate on loans that members of generation t make to each other contingent on the state of nature that occurs at time $t+1$. In the example we have been using where there are two states of nature (with crop sizes of 1/2 or 3/2), members of generation t might choose to make agreements in which the gross interest rate depends on which state occurs. They can agree at time t that if state 1 ($d(t+1) = 1/2$) occurs at time $t+1$, then the gross interest rate will be r^1 , and that if state 2 occurs at time $t+1$, then the gross interest rate will be r^2 .

The asset described in the above paragraph offers a state contingent payout of r^1 in state 1 or r^2 in state two in time $t+1$ for a payment of one unit of the good at time t. Figure X.2 shows one way to illustrate the payouts of a private bond that offers a gross interest rate of 2/3 in state one and a gross interest rate of 5/3 in state 2.

<Insert Figure X.2 here>

Definition 4 *A pure state-i contingent bond at time t pays out one unit of time $t+1$ good if state i occurs at time $t+1$ and nothing if any other state occurs.*

Let $\ell^{hi}(t)$ be the quantity of pure state-i contingent bonds bought by member h of generation t, each of which will pay one unit of the time $t+1$ good if state i occurs at time $t+1$ and zero units of the good if any other state occurs. Let $r^i(t)$ be the time t price of one pure state-i contingent bond that pays off one unit of the time $t+1$ good if and only if state i occurs.

Any security can be made up of a combination of pure state contingent bonds. The bond illustrated in Figure X.2 can be constructed from two pure state contingent bonds: $\ell^{h1}(t) = 2/3$ and $\ell^{h2}(t) = 5/3$. Since we claimed that the bond in Figure X.2 was sold for one unit of the time t

good, that tells us something about the prices of the pure state contingent bonds. The total price of the two pure state contingent bonds adds up to one so

$$1 = r^1(t) \cdot \ell^{h1}(t) + r^2(t) \cdot \ell^{h2}(t).$$

The bond in Figure X.2 is comprised of a linear combination of two pure state contingent bonds.

One unit of time t good will purchase a pure state 1 contingent bond that will pay $1/r^1(t)$ in state one in period $t+1$ or a pure state 2 contingent bond that will pay$1/r^2(t)$ in state two. Both of these points are shown in Figure X.3. Any linear combination of these two bonds (a mixed security) will also cost one unit of the time t good. The bond in Figure X.2 is one such linear combination and is shown at point A. A member of generation t can also purchase a bond that gives a guarenteed return in period $t+1$, the same return in every state. This bond is found where the 45 degree line crosses the line of all linear combinations of the pure state contingent bonds. Point B illustrates this combination in Figure X.3. Note that since buying one unit of each of the two pure state contingent bonds (which garuntees a return of one unit of the time $t+1$ good, independent of which state occurs) costs $r^1(t) + r^2(t)$, then one unit of the time t good will buy a sure return of

$$1/(r^1(t) + r^2(t)).$$

This quantity, $1/(r^1(t) + r^2(t))$ is the risk free gross interest rate.

<Insert Figure X.3 here>

Exercise 1.3.1 *Exercise X.2 There are two states of nature possible for each time period. At time t, the prices on time $t+1$ pure state contingent bonds are$r^1(t) = 1/2$ and $r^2(t) = 1/3$. Graph the set of all mixed security that will cost one unit of time t good. What is the risk free gross interest rate at time t?*

1.4 Optimization under uncertainty

A member of generation t makes decisions over actions at time t that determine consumption at time t and consequences at time $t+1$. The consequences at time $t+1$ that are of interest to us are consumptions at time $t+1$ in each of the states that are possible at time $t+1$. For example, suppose that a member of generation t knows that there are two possible states of the world at time $t+1$ (state 1 where $d(t+1) = 1/2$ and state

2 where $d(t+1) = 3/2$). Otherwise, this economy is the same as example economy 1 in Chapter VI, with

$$N(t) = 100, A = 100, \omega_t^h = [2, 1].$$

We will delay the definition of the utility functions for a moment.

The budget constraints that this individual faces is

$$c_t^h(t) = \omega_t^h(t) - r^1(t) \cdot \ell^{h1}(t) - r^2(t) \cdot \ell^{h2}(t) - p(t) \cdot a^h(t),$$

(X.1a) when young, and either

$$c_t^h(t+1) = \omega_t^h(t+1) + \ell^{h1}(t) + [p^1(t+1) + d^1(t+1)] \cdot a^h(t),$$

(X.1b) if state 1 occurs when old, or

$$c_t^h(t+1) = \omega_t^h(t+1) + \ell^{h2}(t) + [p^2(t+1) + d^2(t+1)] \cdot a^h(t),$$

(X.1c) if state 2 occurs when old. $r^i(t)$ is the time t price of a pure state-i contingent bond. $p^i(t+1)$ is the time $t+1$ price level if state i occurs. Any choice of an $\ell^{h1}(t)$, an $\ell^{h2}(t)$, and an $a^h(t)$ is one action at time t. Each action will result in a pair of consequences which are consumptions of

$$c_t^h = \begin{Bmatrix} [c_t^h(t), c_t^{h1}(t+1)] \text{ in state } 1 \\ [c_t^h(t), c_t^{h2}(t+1)] \text{ in state } 2 \end{Bmatrix}$$

For the example economy 1, we substitute in the values for the endowments and crops and the budget constraints are

$$c_t^h(t) = 2 - r^1(t) \cdot \ell^{h1}(t) - r^2(t) \cdot \ell^{h2}(t) - p(t) \cdot a^h(t),$$

when young and either

$$c_t^{h1}(t+1) = 1 + \ell^{h1}(t) + [p^1(t+1) + 1/2] \cdot a^h(t),$$

or

$$c_t^{h2}(t+1) = 1 + \ell^{h2}(t) + [p^2(t+1) + 3/2] \cdot a^h(t)$$

when old.

Solving equations (X.1b) and (X.1c) for $\ell^{hi}(t)$ and substituting these into equation (X.1a), we get a single budget constraint of the form

$$\begin{aligned} c_t^h(t) \quad & +r^1(t) \cdot c_t^{h1}(t+1) + r^2(t) \cdot c_t^{h2}(t+1) \\ = \quad & \omega_t^h(t) + (r^1(t) + r^2(t)) \cdot \omega_t^h(t+1) \\ & -a^h(t) \cdot \{p(t) - r^1(t) \cdot \left[p^1(t+1) + d^1(t+1)\right] \\ & -r^2(t) \cdot \left[p^2(t+1) + d^2(t+1)\right]\}. \end{aligned}$$

Equation (X.2) is a version of the single budget constraint that we used earlier (equation (IV.3)). Compare the location of the interest rate in equation (IV.3) and the pure state-i contingent loan prices in this equation. Recall that $1/(r^1(t) + r^2(t))$ equals the sure return on an investment of one unit of the time t good and note where the perfectly foreseen interest rate, $r(t)$, occurs in equation (IV.3). We want to find the sign of the part of equation (X.2) that is in the curly brackets. To do that we need to first discuss utility maximization under uncertainty.

In its most general form, a utility function for individual h will depend on the consequences of the actions of individual h, so that each action (each choice of an $\ell^{h1}(t)$, an $\ell^{h2}(t)$, and an $a^h(t)$) will yield a utility of the form

$$u_t^h = u_t^h(c_t^h(t), c_t^{h1}(t+1), c_t^{h2}(t+1)).$$

Individual h's utility as a result of each action depends on the consumption when young that is a result of that action (and is known) and the consumptions that will result in each of the possible states of nature. In the example there are only two states of nature, so there are only three variables in the utility function. If there were more states of nature, there would be more variables in the utility function: one variable for consumption when young and one variable for each of the states that could occur when old.

A utility maximizing individual chooses the action (or is indifferent amoung the set of actions) which gives the highest utility for the consequences that will result from that action. The set of feasible actions is, as before, restricted by the budget constraints. For an individual facing the above budget constraints, the prices on the state contingent bonds, $r^1(t)$ and $r^2(t)$, and the time $t+1$ land prices, $p^1(t+1)$ and $p^2(t+1)$, are known. Since these are known, the consumptions that will result from any choice of $\ell^{h1}(t)$, $\ell^{h2}(t)$, and $a^h(t)$ are known. Given the prices on the state contingent bonds and land prices, it is easy to calculate the consequences of each action.

To choose actions that maximize the utility of the consequences of these actions, an agent will want to choose combinations of state contingent loans and land purchases to maximize the budget constraint and then to choose utility maximizing consumption allocations on that budget constraint. To maximize the budget constraint, each member of generation t tries to get the right hand side of equation (X.2) as large as possible. Everything on the right hand side of equation (X.2) is given except the quantity of land to be bought, $a^h(t)$. The choice of the quantity of land depends on the sign of the stuff inside the curly brackets, or if

$$p(t) \underset{>}{\overset{<}{=}} r^1(t) \cdot [p^1(t+1) + d^1(t+1)] + r^2(t) \cdot [p^2(t+1) + d^2(t+1)].$$

This follows as it did in the perfect foresight case. If $p(t)$ is greater than the right hand side, then land is multiplied by a positive number in equation (X.2) and any land purchases lowers the budget constraint. In that case, no land will be purchased. No arbitrage is possible because we are not allowing private citizens to sell and short. If the price $p(t)$ is less than the right hand side, then each agent will want to borrow as much as possible (an infinite amount) on the loan market and buy an infinite amount of land. Since the amount of land is finite, this cannot be an equilibrium. The price of land, $p(t)$, and the right hand side of the equation must be equal.

One unit of the time t good put into land offers a return of

$$[p^1(t+1) + d^1(t+1)]/p(t)$$

in state one and

$$[p^2(t+1) + d^2(t+1)]/p(t)$$

in state two. We can plot this state contingent return on land on a graph like the one in Figure X.3. This is done in Figure X.4. The line shows the linear combinations of the two pure state contingent private bonds and points A, B, and C show three possible points for the state contingent payouts on land. Point A, with the payout inside the linear combination of the private bonds is the case where $p(t)$ is greater than the right hand side in equation (X.4). Point B, with the payout on land outside the linear combination of the payouts on the private bonds, is the case where $p(t)$ is less than the right hand side in equation (X.4). Case C, on the linear combination, is when $p(t)$ equals the right hand side. In this last case, there is a combination of the pure state contingent private bonds which will exactly match the returns, in each state of nature, on the land.

<Insert Figure X.4>

Note point D on Figure X.4. One can borrow one unit of the time t good for a promise to pay, in each state, a smaller amount than will be generated by using that one unit of the time t good to purchase land (point B shows the returns on land). The line from point D to point B is a 45 degree line and represents a guaranteed return from borrowing one unit of the time t good and using that to buy land. One would want to borrow an infinite amount of the time t good at these rates and purchase an infinite amount of land. For each unit of the time t good borrowed, it is possible to make the same, positive, guaranteed profit. As mentioned above, in an economy with a finite amount of land, we cannot have an equilibrium in which each individual wishes to purchase an infinite amount of land. The land market is not clearing.

The point of this discussion is that the part of equation X.2 in the curly brackets must be equal to zero. That being the case, the budget constraint becomes

$$\begin{aligned} c_t^h(t) \qquad & +r^1(t) \cdot c_t^{h1}(t+1) + r^2(t) \cdot c_t^{h2}(t+1) \\ = \quad & \omega_t^h(t) + (r^1(t) + r^2(t)) \cdot \omega_t^h(t+1), \end{aligned}$$

and only the prices on the pure state contingent bonds matter in determining the budget constraint.

In the beginning of this chapter, we discussed probability and how likely it is that any given state of nature (outcomes of a toss of a coin, for example) will occur. We also noted that individuals can have subjective (private) probabilities on the likelihood of the occurance of states of nature. The utility function in equation (X.3) seems to say nothing about the likelihood of any state of nature occuring. This does not mean that such considerations may not be included in the utility function. Since the form of the utility function is not given, it may very well include, in the way it internally weighs consumption in the different states of nature, subjective estimates of the likelihood of each state of nature occuring. We now make the existence of subjective probabilities more explicit.

We want to be able to write our general utility function (equation (X.3)) in the form

$$u_t^h = \pi^{h1}(t) \cdot v(c_t^h(t), c_t^{h1}(t+1)) + \pi^{h2}(t) \cdot v(c_t^h(t), c_t^{h2}(t+1)).$$

This utility function is comprised of a linear combination of a subutility function, v$(\cdot,\cdot)$, which is evaluated at the consumption consequences of each state of nature. The values of the subutility function at the two states of nature are weighted by the numbers π^{h1}(t) and π^{h2}(t), which we choose to normalize to sum to one. There are only two weights in this two state of nature case. If there were more states of nature, there would need to be more weights. We want to be able to interpret the π^{hi}(t) weights as subjective probabilities and speak of the weighted average of the subutilities over the different states of nature as the expected utility of a given action.

In the appendix to this chapter, we give the four axioms that Kenneth Arrow (1970) required to get an expected utility function such that a preference ordering over actions could be ranked by such a function. In this expected utility function, the subutility function of the consequences of each actions in each state of nature and the subjective probabilities of the states of nature are simultaneously determined. The expected utility function that results gives the required ordering on actions.

Assume that our example economy has an expected utility function with a subutility function of

$$v(c_t^h(t), c_t^{hi}(t+1)) = c_t^h(t) \cdot [c_t^{hi}(t+1)]^\beta,$$

with $\beta < 1$, for all t, and when we do examples we will let the subjective probabilities for each of the two states be

$$\pi^{h1}(t) = .4 \text{ and } \pi^{h2}(t) = .6,$$

for all t. It is quite possible that subjective probabilities would be different for different time periods, but for simplicity we will assume they are the same in each time period. We will also assume that the subutility functions and the subjective probabilities are the same for all individuals in all generations. We can then drop the "h" notation from the subjective probabilities.

The optimization problem for individual h of generation t is to maximize

$$u_t^h = \pi^{h1}(t) \cdot c_t^h(t) \cdot c_t^{h1}(t+1)^\beta + \pi^{h2}(t) \cdot c_t^h(t) \cdot c_t^{h2}(t+1)^\beta,$$

subject to the constraint that

$$\begin{aligned} c_t^h(t) + r^1(t) \cdot c_t^{h1}(t+1) + r^2(t) \cdot c_t^{h2}(t+1) \\ = w_t^h(t) + (r^1(t) + r^2(t)) \cdot w_t^h(t+1). \end{aligned}$$

Recall that an additional constraint for our economy, which is really an arbitrage condition, is that

$$p(t) = r^1(t) \cdot [p^1(t+1) + d^1(t+1)] + r^2(t) \cdot [p^2(t+1) + d^2(t+1)].$$

As before, we want to express the solution to this optimization problem as a saving functions, where

$$s^h(t) = w_t^h(t) - c_t^h(t).$$

We proceed as we did before with finding a savings function and first solve equation (X.7) for time t consumption and then substitute this expression into the utility function, equation (X.6). The two remaining choice variables are consumption in the two states of the world in period $t+1$. We find the point where the derivatives of the utility function with the budget constraint included are equal to zero. From these first order conditions, we find the savings function.

The budget constraint becomes

$$c_t^h(t) = w_t^h(t) + (r^1(t) + r^2(t)) \cdot w_t^h(t+1) - r^1(t) \cdot c_t^{h1}(t+1) + r^2(t) \cdot c_t^{h2}(t+1).$$

Substituting the right hand side of this equation into the utility function given above as equation (X.6) results in

$$\begin{aligned} u_t^h = \quad & [\pi^{h1}(t) \cdot c_t^{h1}(t+1)^\beta + \pi^{h2}(t) \cdot c_t^{h2}(t+1)^\beta] \cdot \\ & [w_t^h(t) + (r^1(t) + r^2(t)) \cdot w_t^h(t+1) - r^1(t) \cdot c_t^{h1}(t+1) + r^2(t) \cdot c_t^{h2}(t+1)]. \end{aligned}$$

To find the utility maximizing point on the budget constraint we find the derivatives

$$\begin{aligned} \partial u_t^h / \partial c_t^{h1} = \quad & -r^1(t) \cdot [\pi^{h1}(t) \cdot c_t^{h1}(t+1)^\beta + \pi^{h2}(t) \cdot c_t^{h2}(t+1)^\beta] \\ + \quad & [w_t^h(t) + (r^1(t) + r^2(t)) \cdot w_t^h(t+1) - r^1(t) \cdot c_t^{h1}(t+1) + r^2(t) \cdot c_t^{h2}(t+1)] \cdot \\ & \beta \cdot \pi^{h1}(t) \cdot c_t^{h1}(t+1)^{(\beta-1)} \\ = \quad & 0, \end{aligned}$$

and

$$\begin{aligned} \partial u_t^h / \partial c_t^{h2} = \quad & -r^2(t) \cdot [\pi^{h1}(t) \cdot c_t^{h1}(t+1)^\beta + \pi^{h2}(t) \cdot c_t^{h2}(t+1)^\beta] \\ + \quad & [w_t^h(t) + (r^1(t) + r^2(t)) \cdot w_t^h(t+1) - r^1(t) \cdot c_t^{h1}(t+1) + r^2(t) \cdot c_t^{h2}(t+1)] \cdot \\ & \beta \cdot \pi^{h2}(t) \cdot c_t^{h2}(t+1)^{(\beta-1)} \\ = \quad & 0, \end{aligned}$$

These first order conditions can be solved (with a substitution from the budget constraint, to get the function,

$$c_t^h(t) = [w_t^h(t) + (r^1(t) + r^2(t)) \cdot w_t^h(t+1)]/(1+\beta).$$

This gives us a savings function for individual h of

$$\begin{aligned} s^h(t) &= w_t^h(t) - c_t^h(t) \\ &= [\beta/(1+\beta)] \cdot w_t^h(t) - [(r^1(t) + r^2(t))/(1+\beta)] \cdot w_t^h(t+1). \end{aligned}$$

Exercise 1.4.1 *Complete the derivation of the above savings function from the first order conditions.*

Exercise 1.4.2 *Find the savings function for the above economy, but with a subutility function of*

$$v(c_t^h(t), c_t^{hi}(t+1)) = c_t^h(t)^\beta + c_t^{hi}(t+1)^\beta,$$

for $\beta < 1$ *and* $i = 1, 2$.

Exercise 1.4.3 *Find the savings function for an economy where all the environmental variables except the endowment when old are the same in every period. There are two states for endowment when old:*

$$w_t^h(t+1) = \begin{Bmatrix} \frac{1}{2} \text{ in state } 1 \\ \frac{3}{2} \text{ in state } 2 \end{Bmatrix}$$

Use the utility function in equation (X.6).

The savings function in equation (X.9) gives us the total quantity of savings that individual h of generation t will want to make at time t. It does not tell us how much of that savings will be allocated towards consumption in state one or consumption in state two. We can, however, find savings functions for each state of nature.

Manipulating the first order conditions, we find a relationship between consumption in state one and consumption in state two. This relationship is

$$\beta \cdot \pi^{h1}(t) \cdot c_t^{h1}(t+1)^{(1-\beta)}/r^1(t) = \beta \cdot \pi^{h2}(t) \cdot c_t^{h2}(t+1)^{(1-\beta)}/r^2(t),$$

and we can solve for state two consumption as a function of state one consumption and have an expression of

$$c_t^{h2}(t+1) = c_t^{h1}(t+1) \cdot [r^1(t) \cdot \pi^1(t)/r^2(t) \cdot \pi^2(t)]^{\frac{1}{1-\beta}}.$$

In completing the derivation in Exercise X.3, one would have used the result that

$$\beta \cdot c_t^h(t) = r^1(t) \cdot c_t^{h1}(t+1) + r^2(t) \cdot c_t^{h2}(t+1).$$

From equation (X.8), we have the time t consumption in terms of the exogenous variables. From equation (X.10), we have state two consumption as a function of state one consumption. Substituting those into equation (X.11), we find state one consumption as a function of the (individually) exogenous variables. The savings that is required to finance that desired state one consumption is equal to the difference between desired consumption in state one at time $t+1$ and time $t+1$ endowment evaluated at the price in time t of a pure state-1 contingent bond. This is

$$s^{h1}(t) = r^1(t) \cdot [c_t^{h1}(t+1) - w_t^h(t+1)],$$

which, after the substitutions are made, becomes the somewhat ungainly

$$\begin{aligned} s^{h1}(t) \quad = \quad & r^1(t) \cdot [(\beta/(1+\beta)) \cdot \{[w_t^h(t) + (r^1(t) + r^2(t)) \cdot w_t^h(t+1)]/ \\ & [r^1(t) + r^2(t) \cdot [r^1(t) \cdot \pi^2(t)/r^2(t) \cdot \pi^1(t)]^{\frac{1}{1-\beta}}\} \\ & -w_t^h(t+1)]. \end{aligned}$$

Equation (X.10) can be used for finding the savings function for state two consumption.

Exercise 1.4.4 *Derive completely the above state one savings function. Derive the savings function for state two savings.*

1.5 A time t temporary equilibrium

Assume that all members of all generations t are alike, so that all of them will have the same savings function. We can find the aggregate savings function by adding up all of the generation t savings functions, or, because they are all alike, by multiplying by the population size. As before, the aggregate savings is

$$S(t) = N(t) \cdot s^h(t),$$

and aggregate savings for states one and two are

$$S(t) = N(t) \cdot s^{hi}(t),$$

for $i = 1, 2$.

Since all members of generation t are identical, there are no opprotunities for intragenerational trade. No pure state-1 contingent bonds are bought or sold. All of the savings goes through the purchase of land. The equilibrium condition that this gives is

$$S(t) = S(r^1(t), r^2(t)) = p(t) \cdot A.$$

The arbitrage condition, equation (X.4) under equality, lets us replace p(t) with

$$p(t) = r^1(t) \cdot [p^1(t+1) + d^1(t+1)] + r^2(t) \cdot [p^2(t+1) + d^2(t+1)].$$

That gives us

$$S(r^1(t), r^2(t)) = \{r^1(t) \cdot [p^1(t+1) + d^1(t+1)] + r^2(t) \cdot [p^2(t+1) + d^2(t+1)]\} \cdot A,$$

which is one equation in two unknowns, r (t) and r (t), once the time $t+1$ prices of the land are given. As in earlier chapters, we wish to find the time t temporary equilibria for this model. In this setting, we can define a time t temporary equilibrium.

Definition 5 *A time t temporary equilibrium in an economy with two states of nature for crops is a price of land, $p(t)$, and prices of pure state-i contingent bonds. $r^i(t)$, $i = 1, 2$ for a given set of time $t+1$ state contingent prices of land, $p^i(t+1)$, $i = 1, 2$ (the environmental variables are assumed to be given).*

We cannot determine the prices of the pure state-i contingent bonds with the current, completely aggregate framework because we do not have enough restrictions on the system to define these prices. Recall that we can separate savings by states of nature and can write a savings function for each state. In an equilibrium, the desired aggregate savings for each state is equal to the returns that the available assets generate if that state occurs times the pure state-i contingent bond price for that state. From this equilibrium condition, we get the two equations

$$S^i(r^1(t), r^2(t)) = r^i(t) \cdot [p^i(t+1) + d^i(t+1)] \cdot A,$$

for $i = 1$ and 2. The two unknowns, of the two equation system one gets when solving for a time t temporary equilibrium, are the prices for the pure state-i contingent bonds, $r^i(t)$, $i = 1$ and 2. The current, time t. price of land, given the state-i prices of land in time$t+1$, is found from the arbitrage condition at equality (equation (X.12)).

Exercise 1.5.1 *Consider the example one economy where $N(t) = 100$, $A = 100$, $w_t^h = [2, 1]$, $d^1(t+1) = 1/2$ and $d^2(t+1) = 3/2$, the utility functions are the ones given in equation (X.6) with $\beta = 1/2$ and $\pi^1(t) = .4$ and $\pi^2(t) = .6$. Find the time t temporary equilibrium for this economy when $p^1(t+1) = 1$ and $p^2(t+1) = 2$. (Hint: First find the $r^i(t)$'s and then find $p(t)$.)*

Assume an economy that is identical to the one given in exercise X.7, except the prices that will occur in time $t+1$ are $p^1(t+1) = 1/2$ and $p^2(t+1) = 1$. We can go about solving the time t temporary equilibrium by graphing the sets of $r^1(t)$ and $r^2(t)$ pairs that solve equation (X.13) for states one and for state two. A pair of prices for the pure state-i contigent bonds that form a time t temporary equilibrium can be found at the intersection of these two lines. Figure X.5 shows the plots of the sets of pure state-i contingent prices for each state of nature for this economy.

<Insert Figure X.5 here>

In Figure X.5, the curve marked S1 shows the pairs of $r^1(t)$ and $r^2(t)$ that solve equation (X.13) for state one. The curve marked S2 shows the pairs for state two. The solution for the time t temporary equilibrium, with $r(t) = .127$ and r(t) = .143, is at the intersection of the two curves. The price of land at time t is found from equation (X.12) and is equal to

$$p(t) = .127 \cdot (1/2 + 1/2) + .143 \cdot (1 + 3/2) = .484.$$

The price for a combination of pure state-i contingent bonds that will always give one unit of the good at time $t+1$ is $.127 + .143 = .270$.

Notice that in a time t temporary equilibrium the time t price of land does not depend in any manner on the state of nature that prevails at time t, it depends on the states of nature that can prevail at time $t+1$. Nowhere in the above discussion did the state of nature at time t appear in the equations determining the price of land at time t. The supply of land is fixed at A units in each state of nature. The old will sell all of their land holdings in either state of nature. Only the demand for land at time t determines its price. The demand for land comes from the young of generation t. Their demand is determined by the prices and crops that they foresee in each state of nature at time $t+1$ and the subjective probabilities they have about the likelihood that each of those states of nature will occur.

If the young at time t have enough information about the economy, they will follow the logic of the above paragraph and realize that the price of land which will occur at time $t+1$ will be independent of the state of nature that occurs at time $t+1$. Since the price of land that occurs at time $t+1$ is independent of the state of nature at time $t+1$, there can be only one price for land in that period. In exercise X.7, we should not have had two different prices for land in the two states of the world,since the independence of the price of land at time t to the state of the world at time t means that $p^1(t+1)$ equals $p^2(t+1)$. We can drop the superscript from the price of land at time $t+1$ and merely use $p(t+1)$ to indicate the one price that is expected to exist in that period.

1.6 A stationary equilibrium

We have been quite specific in the construction of our model. Each period t is like any other period in terms of population, land, endowments, preferences, and in the states of nature that can occur in period $t+1$. The assumption that preferences are the same in each period mean that individuals of every generation t have the same subjective probabilities about the likelihood of each of the two states of nature in time $t+1$. In terms of the economic environment, every period looks like every other period. In addition, we have shown in the previous section that the price of land which occurs in each period is independent of the state of nature that occurs in that period. Consequently, there is only one price of land in each period.

Since the environment is the same in each period, we can look for a stationary equilibrium for this economy. In our economy with uncertainty, a stationary equilibrium is not one in which the consumptions of each generation are, necessarily, the same as the consumptions of every other generation. Instead, a stationary equilibrium is one in which the price of land and the prices of the two pure state-i contingent bonds are the same in each period. The consumptions, when old, of generations t and s can be different if the realizations of the state of nature at times $t+1$ and $s+1$ are different. In a stationary equilibrium, the consumption when young of all generations will be the same since the young of every generation have the same preferences and face the same budget constraints. Only consumption when old can be different and will depend on which of the two (in our examples) states of nature occurs.

Definition 6 *A stationary state equilibrium for an economy with multiple states of nature for the environmental variables is a set of prices of land and of pure state-i contingent bonds,* $[p(t), \{r^i(t)\}]$ *for all states of nature* i*, that are the same for all periods t such that the markets for bonds and land clear each period and all agents maximize their welfare subject to their budget constraints.*

The key point to the definition is that the same set of prices works for all periods. It is a requirement for this kind of equilibrium that every period look just like every other period. In particular, every generation must have the same pattern of preferences as every other generation and the endowments and the crops must be the same. If there is growth of population, changes in the size of crops over time, or generational differences in preferences, we cannot expect to find stationary equilibria. Other, non-stationary equilibria can exist, but they are somewhat more difficult to characterize.

The basic method for finding a stationary equilibrium is the same as for finding the stationary state perfect foresight equilibria that we have been using in previous chapters. A time t temporary equilibrium (an $r^1(t)$, an $r^2(t)$, and a $p(t)$) can be found for each $p(t+1)$. A stationary equilibrium occurs when a p(t+1) generates a p(t) with the same value as that p(t+1). These fixed points (there may be more than one, we have had multiple equilibria before) are the stationary equilibria we desire.

<Insert Figure X.6 here>

For each $p(t+1)$, we find the time t temporary equilibrium that it induces. This is done by using equation (X.13) for each of the states of nature and then finding the pair of prices for the pure state-i contingent bonds. The arbitrage condition gives the time t price of land, $p(t)$, from these prices for the state-i contingent bonds, the time $t+1$ price of land, the crops in each state. Figure X.6 shows three pairs of graphs of the prices forthe pure state-i contingent bonds (the solid is for $p(t+1) = 1$, the dashed for $p(t+1) = 1/2$ and the dotted for $p(t+1) = 3/2$). The intersection of each pair gives the time t temporary equilibrium prices for the pure state-i contingent bonds. From the arbitrage conditions we find the time t price for land in each of these time t temporary equilibria. For example, when $p(t+1)$ equals 1, the time t price is

$$p(t) = .108 \cdot (1 + 1/2) + .137 \cdot (1 + 3/2) = .504.$$

Figure X.6 shows how the prices of the pure state-i contingent bonds are functions of the time $t+1$ price of land (as well as the subutility functions, the subjective probabilities, and the state contingent crop sizes). Concentrating on the time $t+1$ price of land, we can write the price of the pure state-i contingent bond as

$$r^i(t) = r^i(p(t+1)),$$

and the time t price of land as

$$\begin{aligned} p(t) &= r^1(p(t+1)) \cdot (p(t+1) + d^1(t+1)) + r^2(p(t+1)) \cdot (p(t+1) + d^2(t+1)) \\ &= f(p(t+1)). \end{aligned}$$

The function f(·) is conceptually identical to mapping from p($t+1$) to p(t) we found in Chapter VI. The function from $p(t+1)$ to $p(t)$ for the economy we used in Exercise X.7 is shown in Figure X.7. The forty-five degree line, the dashed line in Figure X.7) shows all points for which $p(t+1)$ equals $p(t)$. The stationary equilibrium for this economy is where the forty-five degree line crosses $f(\cdot)$. That occurs at a price of $p(t) = p(t+1) = .4628$, which is indicated by the dotted line in Figure X.7.

<Insert Figure X.7 here>

The uniqueness of a stationary equilibrium (given that the environmental variables are the same in each period) can be shown by seeing what happens if the time t price of land is something other than the stationary value. (We are following exactly the same process as we did in chapter VI.) A price for land at time t below the stationary price requires a time $t+1$ price that is even lower. At some time $t+n$, for an n that is not very large, the price of land becomes negative. If the price of land at time t is greater than the stationary value, the price explodes and eventually becomes undefined. In the economy we have been studying, the stationary equilibrium is unique.

Exercise 1.6.1 *For the stationary equilibrium of the example economy solved in the above section, find the consumption, when young, for all members of all generations. Find the consumption, when old, for all members of the generations for which state 1 occurs when they are old. Find the consumption, when old, for members of the generations for which state 2 occurs when they are old. Calculate the expected utility of all members of all generations t, for* $t \geq 1$.

1.7 Reprise

Extending the overlapping generations model to handle assets that give uncertain returns is fairly simple. The most important modification involves the characterization of the utility function as one where expected utility with subjective probabilities of the likelihood of each state of nature. Once this characterization of the utility function is done, solving the model for an equilibrium proceeds as it did before. We encounter increased dimensionality in all of the calculations because of the addition of states of nature to the two periods of life, but that is not difficult to deal with. Stationary equilibirum exist and are unique for the economies we considered.

The reader might have noticed that we set the beta parameter of the utility functions to be less than one in all of the example economies. A particular difficulty arises if β equals 1. The example expected utility function we have been using is

$$u_t^h = \pi^{h1}(t) \cdot c_t^h(t) \cdot c_t^{h1}(t+1)^\beta + \pi^{h2}(t) \cdot c_t^h(t) \cdot c_t^{h2}(t+1)^\beta,$$

and with equal to 1, this utility function becomes

$$u_t^h = c_t^h(t) \cdot [\pi^{h1}(t) \cdot c_t^{h1}(t+1) + \pi^{h2}(t) \cdot c_t^{h2}(t+1)].$$

The portion in square brackets is linear over states of nature. This utility function will give linear indifference curves over the two states of nature. A linear budget constraint over the two states of nature will give us corner solutions, so only one contingent bond is purchased, or (if the probabilities and the prices of the pure state-i contingent bonds are the same) an infinite number of solutions (the whole indifference curve).

It is perfectly possible that individuals have expected utility functions with subutility functions with a β equal to one. In that case, calculus is not very helpful for solving the optimization problem and linear programing techniques are necessary. Since we have restricted ourselves to using calculus, it was necessary to restrict the value of β to be less than one.