Extra stuff for Hansens RBC model

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1 Additional material for the RBC model

.Constructing an RBC model (log-linear version)

- Write out model
  - Optimization problem of households
    - First order conditions
    - Budget constrains
  - Optimization problem of firms
    - First order conditions
    - Budget constrains
  - Optimization problem of other agents
    - Government
    - Financial intermediaries
  - Aggregation conditions
  - Equilibrium conditions
- Find stationary state
- Log-linearize model around stationary state
- Solve linear version for linear plans
- Analyze second moments of model

.Hansen's basic model

- Robinson Crusoe maximizes the discounted utility function

$$\max \sum_{t=0}^{\infty} \beta^t u(c_t, l_t)$$

- The specific utility functions

$$u(c_t, 1-h_t) = \ln c_t + A \ln(1-h_t)$$

with $A > 0$.

- The production function is

$$f(\lambda_t, k_t, h_t) = \lambda_t k_t^\theta h_t^{1-\theta}$$

- λ_t is a random technology variable that follows the process

$$\lambda_{t+1} = \gamma \lambda_t + \varepsilon_{t+1}$$

for $0 < \gamma < 1$. ε_t iid, positive, bounded above, $E\varepsilon_t = 1 - \gamma$.

  - $\Longrightarrow E\ \lambda_t$ is 1 and $\lambda_{t+1} > 0$.

Hansen's basic model (continued)

- Capital accumulation follows the process

$$k_{t+1} = (1-\delta)k_t + i_t$$

- The feasibility constraint is

$$f(\lambda_t, k_t, h_t) \geq c_t + i_t$$

Log linear version of Hansen's model

- The five equations of the Hansen model are (adjusted)

$$\begin{aligned}
1 &= \beta E_t \left[\frac{C_t}{C_{t+1}} \left(r_{t+1} + (1-\delta) \right) \right] \\
AC_t &= (1-\theta)(1-H_t) \frac{Y_t}{H_t} \\
C_t &= Y_t + (1-\delta)K_t - K_{t+1} \\
Y_t &= \lambda_t K_t^\theta H_t^{1-\theta} \\
r_t &= \theta \frac{Y_t}{K_t}
\end{aligned}$$

Log-linearization (Uhlig's method)

- Define the log difference of each variable as

$$\widetilde{X}_t = \ln X_t - \ln \overline{X}$$

- Then the original variable is

$$X_t = \overline{X} e^{\widetilde{X}_t}$$

- Substitute this into each equation (here a production function)

$$Y_t = \lambda_t K_t^{\theta} H_t^{1-\theta}$$

- becomes

$$\overline{Y} e^{\widetilde{Y}_t} = e^{\widetilde{\lambda}_t} \overline{K}^{\theta} e^{\theta \widetilde{K}_t} \overline{H}^{(1-\theta)} e^{(1-\theta)\widetilde{H}_t}$$

Log-linearization (Uhlig's method) continued

- Because $\overline{Y} = \overline{K}^{\theta} \overline{H}^{(1-\theta)}$, this simplifies to

$$e^{\widetilde{Y}_t} = e^{\widetilde{\lambda}_t + \theta \widetilde{K}_t + (1-\theta)\widetilde{H}_t}$$

  and, using the approximation $e^{\widetilde{X}_t} \approx 1 + \widetilde{X}_t$, if $\widetilde{X}_t$ is small, this equals

$$1 + \widetilde{Y}_t = 1 + \widetilde{\lambda}_t + \theta \widetilde{K}_t + (1-\theta) \widetilde{H}_t$$

  and

$$\widetilde{Y}_t = \widetilde{\lambda}_t + \theta \widetilde{K}_t + (1-\theta) \widetilde{H}_t$$

.

- Recall the direct method: we got

$$\frac{Y_t}{\overline{Y}} + 1 \approx \frac{\lambda_t}{\overline{\lambda}} + \frac{\theta K_t}{\overline{K}} + \frac{(1-\theta) H_t}{\overline{H}}$$

- write as

$$\frac{\overline{Y} e^{\widetilde{Y}_t}}{\overline{Y}} + 1 \approx \frac{\overline{\lambda} e^{\widetilde{\lambda}_t}}{\overline{\lambda}} + \frac{\overline{K} \theta e^{\widetilde{K}_t}}{\overline{K}} + \frac{\overline{H} (1-\theta) e^{\widetilde{H}_t}}{\overline{H}}$$

- or

$$e^{\widetilde{Y}_t} + 1 \approx e^{\widetilde{\lambda}_t} + \theta e^{\widetilde{K}_t} + (1-\theta) e^{\widetilde{H}_t}$$

- which is approximately

$$\left(1 + \widetilde{Y}_t\right) + 1 \approx \left(1 + \widetilde{\lambda}_t\right) + \theta \left(1 + \widetilde{K}_t\right) + (1-\theta) \left(1 + \widetilde{H}_t\right)$$

- or

$$\widetilde{Y}_t \approx \widetilde{\lambda}_t + \theta \widetilde{K}_t + (1-\theta) \widetilde{H}_t$$

Another example

- Consider (a first order condition of the Hansen model)

$$1=\beta E_t\left[\frac{C_t}{C_{t+1}}\left(r_{t+1}+(1-\delta)\right)\right]$$

$$\begin{aligned}1 &= \beta E_t\left[\frac{\overline{C}e^{\widetilde{C}_t}}{\overline{C}e^{\widetilde{C}_{t+1}}}\overline{r}e^{\widetilde{r}_{t+1}}+(1-\delta)\frac{\overline{C}e^{\widetilde{C}_t}}{\overline{C}e^{\widetilde{C}_{t+1}}}\right]\\ &= \beta E_t\left[\overline{r}e^{\widetilde{C}_t-\widetilde{C}_{t+1}+\widetilde{r}_{t+1}}+(1-\delta)e^{\widetilde{C}_t-\widetilde{C}_{t+1}}\right]\\ &\approx \beta\left(\overline{r}E_t\left[1+\widetilde{C}_t-\widetilde{C}_{t+1}+\widetilde{r}_{t+1}\right]+(1-\delta)\left[1+\widetilde{C}_t-\widetilde{C}_{t+1}\right]\right)\\ &= E_t\left[1+\widetilde{C}_t-\widetilde{C}_{t+1}+\beta\overline{r}\widetilde{r}_{t+1}\right],\end{aligned}$$

  or (after cancelling the 1's and cleaning up the expections)

$$0\approx\widetilde{C}_t-E_t\widetilde{C}_{t+1}+\beta\overline{r}E_t\widetilde{r}_{t+1}$$

Another example

- The household budget constraint

$$C_t=Y_t+(1-\delta)K_t-K_{t+1}$$

- Becomes

$$\overline{C}e^{\widetilde{C}_t}=\overline{Y}e^{\widetilde{Y}_t}+(1-\delta)\overline{K}e^{\widetilde{K}_t}-\overline{K}e^{\widetilde{K}_{t+1}}$$

- and this is approximately

$$\overline{C}\left(1+\widetilde{C}_t\right)=\overline{Y}\left(1+\widetilde{Y}_t\right)+(1-\delta)\overline{K}\left(1+\widetilde{K}_t\right)-\overline{K}\left(1+\widetilde{K}_{t+1}\right)$$

- Given the stationary state, this reduces to

$$\overline{C}\widetilde{C}_t=\overline{Y}\widetilde{Y}_t+(1-\delta)\overline{K}\widetilde{K}_t-\overline{K}\widetilde{K}_{t+1}$$

Another example

- The second first order condition from the Hansen model

$$AC_t=(1-\theta)(1-H_t)\frac{Y_t}{H_t}$$

- Bring H_t over to the left hand side and expand the right hand side

$$AC_tH_t=(1-\theta)Y_t-(1-\theta)H_tY_t$$

- Put in the log-linear expression

$$A\overline{CH}e^{\widetilde{C}_t+\widetilde{H}_t} = (1-\theta)\,\overline{Y}e^{\widetilde{Y}_t} - (1-\theta)\,\overline{HY}e^{\widetilde{H}_t+\widetilde{Y}_t}$$

- Approximate

$$\begin{aligned} A\overline{CH}\left(1+\widetilde{C}_t+\widetilde{H}_t\right) &= (1-\theta)\,\overline{Y}\left(1+\widetilde{Y}_t\right) \\ &\quad -(1-\theta)\,\overline{HY}\left(1+\widetilde{H}_t+\widetilde{Y}_t\right) \end{aligned}$$

.

- Using conditions from the stationary state this simplifies to

$$A\overline{CH}\left(\widetilde{C}_t+\widetilde{H}_t\right) = (1-\theta)\,\overline{Y}\widetilde{Y}_t - (1-\theta)\,\overline{HY}\left(\widetilde{H}_t+\widetilde{Y}_t\right)$$

- Rearranging this becomes

$$A\overline{CH}\widetilde{C}_t = (1-\theta)\,\overline{Y}\left(1-\overline{H}\right)\widetilde{Y}_t - \overline{H}\left[(1-\theta)\,\overline{Y}+A\overline{C}\right]\widetilde{H}_t$$

- which can be further simplified (because $\left[(1-\theta)\,\overline{Y}+A\overline{C}\right]\overline{H} = (1-\theta)\,\overline{Y}$) to

$$\frac{A\overline{CH}}{(1-\theta)\,\overline{Y}}\widetilde{C}_t = \left(1-\overline{H}\right)\widetilde{Y}_t - \widetilde{H}_t$$

- and again (because $\frac{A\overline{CH}}{(1-\theta)\overline{Y}} = 1-\overline{H}$) to

$$\widetilde{C}_t = \widetilde{Y}_t - \frac{\widetilde{H}_t}{\left(1-\overline{H}\right)}$$

.How to handle a problematic equation

- An equation of the form

$$Y_t = \sum_{i=0}^{\infty}\beta^i\frac{Z_{t+i}}{1-Z_{t+i}}$$

  does not let you bring the $1-Z_{t+i}$ part over to the other side

- Need to do a number of approximations

$$\begin{aligned} \frac{Z_{t+i}}{1-Z_{t+i}} &= \frac{\overline{Z}e^{\widetilde{Z}_{t+i}}}{1-\overline{Z}e^{\widetilde{Z}_{t+i}}} \approx \frac{\overline{Z}\left(1+\widetilde{Z}_{t+i}\right)}{1-\overline{Z}\left(1+\widetilde{Z}_{t+i}\right)} \\ &= \frac{\overline{Z}}{1-\overline{Z}}\frac{1+\widetilde{Z}_{t+i}}{\left(1-\frac{\overline{Z}}{1-\overline{Z}}\widetilde{Z}_{t+i}\right)} \end{aligned}$$

- But

$$\left(1-\frac{\overline{Z}}{1-\overline{Z}}\widetilde{Z}_{t+i}\right)\approx e^{-\frac{\overline{Z}}{1-\overline{Z}}\widetilde{Z}_{t+i}}$$

.

- So the equation can be written as

$$\begin{aligned}
&= \frac{\overline{Z}}{1-\overline{Z}}\left(1+\widetilde{Z}_{t+i}\right)e^{\frac{\overline{Z}}{1-\overline{Z}}\widetilde{Z}_{t+i}} \\
&= \frac{\overline{Z}}{1-\overline{Z}}\left(1+\widetilde{Z}_{t+i}\right)\left(1+\frac{\overline{Z}}{1-\overline{Z}}\widetilde{Z}_{t+i}\right) \\
&= \frac{\overline{Z}}{1-\overline{Z}}\left(1+\widetilde{Z}_{t+i}+\frac{\overline{Z}}{1-\overline{Z}}\widetilde{Z}_{t+i}+\frac{\overline{Z}}{1-\overline{Z}}\widetilde{Z}_{t+i}\widetilde{Z}_{t+i}\right) \\
&= \frac{\overline{Z}}{1-\overline{Z}}\left(1+\widetilde{Z}_{t+i}+\frac{\overline{Z}}{1-\overline{Z}}\widetilde{Z}_{t+i}\right) \\
&= \frac{\overline{Z}}{1-\overline{Z}}\left(1+\frac{1}{1-\overline{Z}}\widetilde{Z}_{t+i}\right)
\end{aligned}$$

.

- So

$$\begin{aligned}
\overline{Y}\left(1+\widetilde{Y}_t\right) &= \frac{\overline{Z}}{1-\overline{Z}}\sum_{i=0}^{\infty}\beta\left(1+\frac{1}{1-\overline{Z}}\widetilde{Z}_{t+i}\right) \\
&= \frac{\overline{Z}}{\left(1-\overline{Z}\right)(1-\beta)}+\frac{\overline{Z}}{\left(1-\overline{Z}\right)^2}\sum_{i=0}^{\infty}\beta^i\widetilde{Z}_{t+i}
\end{aligned}$$

- stationary state of $Y_t=\sum_{i=0}^{\infty}\beta^i\frac{Z_{t+i}}{1-Z_{t+i}}$ is $\overline{Y}=\frac{\overline{Z}}{\left(1-\overline{Z}\right)(1-\beta)}$, so this becomes

$$\widetilde{Y}_t=\frac{(1-\beta)}{\left(1-\overline{Z}\right)}\sum_{i=0}^{\infty}\beta^i\widetilde{Z}_{t+i}$$