1 Solving matrix quadratic equations

Solving matrix quadratic equations

- We look for a solution to the quadratic equation,

\[ AP^2 - BP - C = 0 \]

- of the form \( P = \Psi \Lambda \Psi^{-1} \)

- where \( \Lambda \) is a matrix of eigenvalues on the diagonal of the form

\[
\Lambda = \begin{bmatrix}
\lambda_1 & 0 & \cdots & \cdots & 0 \\
0 & \lambda_2 & 0 & \cdots & \vdots \\
\vdots & 0 & \ddots & \ddots & \vdots \\
\vdots & \cdots & \cdots & \lambda_{n-1} & 0 \\
0 & \cdots & \cdots & 0 & \lambda_n \\
\end{bmatrix}
\]

- \( \Psi \) is a matrix with the corresponding eigenvectors.

- This way of writing \( P \) gives \( P^2 = \Psi \Lambda \Psi^{-1} \Psi \Lambda \Psi^{-1} = \Psi \Lambda^2 \Psi^{-1} \)

Solving matrix quadratic equations

- The matrices \( A, B, \) and \( C \) of \( AP^2 - BP - C = 0 \) are all \( n \times n \).

- Construct the \( 2n \times 2n \) matrices

\[
D = \begin{bmatrix} B & C \\ I & 0 \end{bmatrix}
\]

and

\[
E = \begin{bmatrix} A & 0 \\ 0 & I \end{bmatrix}
\]

Solving matrix quadratic equations

- Find the solution to the generalized eigenvalue problem for the matrix pair \((D, E)\).

- The solution to this problem is a set of \( 2n \) eigenvalues \( \lambda_k \) and corresponding eigenvectors \( x_k \), such that

\[ Dx_k = Ex_k \lambda_k \]

- Assume that there are at least \( n \) stable eigenvectors, those whose absolute value is less than one.
Order the eigenvalues and their corresponding eigenvectors, so that the \( n \) stable eigenvalues come first.

Solving matrix quadratic equations

- The eigenvectors are columns, so that the matrix \( X \) is

\[
X = \begin{bmatrix}
  x_{1,1} & x_{2,1} & \cdots & x_{2n,1} \\
  x_{1,2} & x_{2,2} & \cdots & x_{2n,2} \\
  \vdots & \vdots & \ddots & \vdots \\
  x_{1,2n} & x_{2,2n} & \cdots & x_{2n,2n}
\end{bmatrix}.
\]

Solving matrix quadratic equations

Partition \( X \) so that

\[
X = \begin{bmatrix}
  X_{11} & X_{21} \\
  X_{12} & X_{22}
\end{bmatrix} =
\begin{bmatrix}
  X_{11} & X_{21} \\
  X_{12} & X_{22}
\end{bmatrix} =
\begin{bmatrix}
  x_{1,1} & x_{2,1} & \cdots & x_{n,1} \\
  x_{1,2} & x_{2,2} & \cdots & x_{n,2} \\
  \vdots & \vdots & \ddots & \vdots \\
  x_{1,n} & x_{2,n} & \cdots & x_{n,n}
\end{bmatrix}
\begin{bmatrix}
  x_{n+1,1} & x_{n+1,2} & \cdots & x_{n+2,1} \\
  x_{n+1,2} & x_{n+1,2} & \cdots & x_{n+2,2} \\
  \vdots & \vdots & \ddots & \vdots \\
  x_{n+1,n+1} & x_{n+1,n} & \cdots & x_{n+2,n}
\end{bmatrix}.
\]

The generalized eigenvalues gives the problem in the form

\[
\begin{bmatrix}
  B & C \\
  I & 0
\end{bmatrix}
\begin{bmatrix}
  X_{11} & X_{21} \\
  X_{12} & X_{22}
\end{bmatrix} =
\begin{bmatrix}
  A & 0 \\
  0 & I
\end{bmatrix}
\begin{bmatrix}
  X_{11} & X_{21} \\
  X_{12} & X_{22}
\end{bmatrix}
\begin{bmatrix}
  \Delta & 0 \\
  0 & \Delta^2
\end{bmatrix}
\]

Multiplying out the matrices on each side gives

\[
\begin{bmatrix}
  BX_{11} + CX_{12} \\
  X^T_{11}
\end{bmatrix}
\begin{bmatrix}
  A X_{11} & A X_{21} \\
  X^T_{11} & X^T_{21}
\end{bmatrix} =
\begin{bmatrix}
  X^T_{11} & X^T_{21}
\end{bmatrix}
\begin{bmatrix}
  \Delta & 0 \\
  0 & \Delta^2
\end{bmatrix}
\]

Solving matrix quadratic equations

- Looking at corresponding partitions, we use

\[
X_{11} = X^T_{12} \Delta^1,
\]

and

\[
BX_{11} + CX_{12} = AX_{11} \Delta^1.
\]
Substituting in $X^{12}\Delta^1$ for $X^{11}$ in the second equation gives

$$BX^{12}\Delta^1 + CX^{12} = AX^{12}\Delta^1\Delta^1,$$

postmultiplying both sides by $(X^{12})^{-1}$ gives

$$BX^{12}\Delta^1 (X^{12})^{-1} + C = AX^{12}\Delta^1\Delta^1 (X^{12})^{-1}.$$

Define $P = X^{12}\Delta^1 (X^{12})^{-1}$.

Then $P^2 = X^{12}\Delta^1\Delta^1 (X^{12})^{-1}$ and, from above,

$$BP + C = AP^2.$$

Solving matrix quadratic equations

Therefore, the solution to the matrix quadratic equation can be found by constructing the matrices $D$ and $E$ and finding the solution to the generalized eigenvalue problem for those matrices as the generalized eigenvector matrix $X$ and the generalized eigenvalue matrix $\Delta$ (ordered appropriately, with the stable eigenvalues first). The matrix $\Delta^1$, contains the eigenvalues and the matrix $X^{12}$ contains the eigenvectors that we use to construct

$$P = X^{12}\Delta^1 (X^{12})^{-1}.$$ 

Solving matrix quadratic equations

Summarized

- To solve: form matrices $D$ and $E$
- Use Matlab program $eig$ in the form

$$[V, D] = eig(A, B)$$

produces a diagonal matrix $D$ of generalized eigenvalues and a full matrix $V$ whose columns are the corresponding eigenvectors so that $A \ast V = B \ast V \ast D$.

- Select the eigenvalues with absolute values less than one
- Select the corresponding eigenvectors
- Use these to make the matrix $X^{11}$, $X^{12}$, and $\Delta^1$.