# 1 A two sector model: accumulation human capital

## 1.1 Model of Uzawa and Lucas

Suppose that there are two forms of accumulation capital that use different production function functions. Physical capital comes from the production of goods and the decision of how much to save of this production. Goods production at time $t$ is

$$K_{t+1} = AK_t(y)^{\theta} H_t(y)^{1-\theta} - C_t + (1-\delta_k) K_t$$

where $K_t(y)$ is the amount of time $t$ capital devoted to goods production and $H_t(y)$ is the amount of time $t$ human capital devoted to goods production. Capital production follows

$$H_{t+1} = BK_t(h)^{\eta} H_t(h)^{1-\eta} + (1-\delta_h) H_t.$$

We assume that both time $t$ physical capital and human capital are fully utilized so that

$$K_t = K_t(y) + K_t(h)$$

and

$$H_t = H_t(y) + H_t(h).$$

To simplify notation, we write

$$H_t(y) = \mu_t H_t$$

so

$$H_t(h) = (1-\mu_t) H_t.$$

Labor market literature indicates that the production of human capital is relatively intensive in human capital and the production of goods is relatively intensive in physical capital, so that

$$\theta > \eta.$$

To make the model as simple as possible and maintain these assumptions, Uzawa and Lucas assume the limit condition that $\eta = 0$. With these assumptions, the model can be written as

$$K_{t+1} = AK_t^{\theta} (\mu_t H_t)^{1-\theta} - C_t + (1-\delta_k) K_t$$

and

$$H_{t+1} = B(1-\mu_t) H_t + (1-\delta_h) H_t.$$

Writing out these equations in per capita terms (here we will not worry about population growth but are keeping in the spirit of the papers) we get

$$k_{t+1} = Ak_t^{\theta} (\mu_t h_t)^{1-\theta} - c_t + (1-\delta_k) k_t$$

and

$$h_{t+1} = B\left(1-\mu_t\right)h_t + \left(1-\delta_h\right)h_t.$$

We assume that individuals want to maximize a CES utility function of the form

$$U_0 = \sum_{t=0}^{\infty} \beta^t \frac{c_t^{1-\alpha}-1}{1-\alpha}$$

and write out the Lagrangean

$$\begin{aligned}\mathcal{L} &= \sum_{t=0}^{\infty} \beta^t \left[\frac{c_t^{1-\alpha}-1}{1-\alpha} + \lambda_t^1 \left(k_{t+1} - Ak_t^{\theta}\left(\mu_t h_t\right)^{1-\theta} + c_t - \left(1-\delta_k\right)k_t\right)\right. \\ &\quad \left. + \lambda_t^2\left(h_{t+1} - B\left(1-\mu_t\right)h_t - \left(1-\delta_h\right)h_t\right)\right].\end{aligned}$$

We get the first order condtions of

$$\begin{aligned}\frac{\partial \mathcal{L}}{\partial c_s} &= c_s^{-\alpha} + \lambda_t^1 = 0 \\ \frac{\partial \mathcal{L}}{\partial k_{t+1}} &= \lambda_t^1 - \beta\lambda_{t+1}^1\left[\theta A k_{t+1}^{\theta-1}\left(\mu_{t+1}h_{t+1}\right)^{1-\theta} - \left(1-\delta_k\right)\right] = 0 \\ \frac{\partial \mathcal{L}}{\partial h_{t+1}} &= -\beta\lambda_{t+1}^1\left(1-\theta\right)\mu_{t+1}Ak_{t+1}^{\theta}\left(\mu_{t+1}h_{t+1}\right)^{-\theta} + \lambda_t^2 - \beta\lambda_{t+1}^2\left[B\left(1-\mu_{t+1}\right) - \left(1-\delta_h\right)\right] = 0 \\ \frac{\partial \mathcal{L}}{\partial \mu_s} &= -\lambda_t^1\left(1-\theta\right)h_t Ak_t^{\theta}\left(\mu_t h_t\right)^{-\theta} + \lambda_t^2 B h_t = 0\end{aligned}$$

and the two budget constraints

$$k_{t+1} = Ak_t^{\theta}\left(\mu_t h_t\right)^{1-\theta} - c_t + \left(1-\delta_k\right)k_t$$

and

$$h_{t+1} = B\left(1-\mu_t\right)h_t + \left(1-\delta_h\right)h_t.$$

The first order conditions can be rewritten as

$$\begin{aligned}-c_t^{-\alpha} &= \lambda_t^1 \\ \lambda_t^1 &= \beta\lambda_{t+1}^1\left[\theta A k_{t+1}^{\theta-1}\left(\mu_{t+1}h_{t+1}\right)^{1-\theta} - \left(1-\delta_k\right)\right] \\ \beta\lambda_{t+1}^1\left(1-\theta\right)\mu_{t+1}Ak_{t+1}^{\theta}\left(\mu_{t+1}h_{t+1}\right)^{-\theta} &= \lambda_t^2 - \beta\lambda_{t+1}^2\left[B\left(1-\mu_{t+1}\right) - \left(1-\delta_h\right)\right] \\ \lambda_t^1\left(1-\theta\right)Ak_t^{\theta}\left(\mu_t h_t\right)^{-\theta} &= \lambda_t^2 B\end{aligned}$$

and further simplified to

$$\begin{aligned}
\left(\frac{c_{t+1}}{c_t}\right)^{\alpha} &= \beta\left[\theta A k_{t+1}^{\theta-1}\left(\mu_{t+1}h_{t+1}\right)^{1-\theta}-(1-\delta_k)\right] \\
\beta\frac{(1-\theta)\,\mu_{t+1}Ak_{t+1}^{\theta}\left(\mu_{t+1}h_{t+1}\right)^{-\theta}}{c_{t+1}^{\alpha}} &= \frac{(1-\theta)\,Ak_t^{\theta}\left(\mu_t h_t\right)^{-\theta}}{c_t^{\alpha}B} \\
&\quad -\beta\left[\frac{(1-\theta)\,Ak_{t+1}^{\theta}\left(\mu_{t+1}h_{t+1}\right)^{-\theta}}{c_{t+1}^{\alpha}B}\right]\left[B\left(1-\mu_{t+1}\right)-(1-\delta_h)\right] \\
\lambda_t^2 &= -\frac{(1-\theta)\,Ak_t^{\theta}\left(\mu_t h_t\right)^{-\theta}}{c_t^{\alpha}B}.
\end{aligned}$$

Let the depreciation rates for capital and human capital be the same. The model is

$$\begin{aligned}
\left(\frac{c_{t+1}}{c_t}\right)^{\alpha} &= \beta\left[\theta A k_{t+1}^{\theta-1}\left(\mu_{t+1}h_{t+1}\right)^{1-\theta}-(1-\delta)\right] \\
\frac{Ak_t^{\theta}\left(\mu_t h_t\right)^{-\theta}}{c_t^{\alpha}} &= \beta\left[B-(1-\delta)\right]\frac{Ak_{t+1}^{\theta}\left(\mu_{t+1}h_{t+1}\right)^{-\theta}}{c_{t+1}^{\alpha}}
\end{aligned}$$

$$k_{t+1} = Ak_t^{\theta}\left(\mu_t h_t\right)^{1-\theta} - c_t + (1-\delta)\,k_t$$

and

$$h_{t+1} = B\left(1-\mu_t\right)h_t + (1-\delta)\,h_t.$$

We will look for an equilibrium in which all the variables grow at the same rate, so we look for cases where $\gamma_t^k = \gamma_t^h = \gamma_t^y = \gamma_t^c = \gamma^*$. In a situation where the growth rates are the same, $\mu_t$ needs to be a constant, $\mu^*$. Write the growth rates of capital and human capital as

$$\begin{aligned}
\gamma_t^k &= \frac{k_{t+1}}{k_t}-1 = Ak_t^{\theta-1}\left(\mu_t h_t\right)^{1-\theta}-\frac{c_t}{k_t}-\delta \\
&= \frac{y_t}{k_t}-\frac{c_t}{k_t}-\delta
\end{aligned}$$

and

$$\gamma_t^h = \frac{h_{t+1}}{h_t}-1 = B\left(1-\mu_t\right)-\delta.$$

we also get

$$\begin{aligned}
\gamma_t^c &= \left(\frac{c_{t+1}}{c_t}-1\right) = \beta^{\frac{1}{\alpha}}\left[\theta A k_{t+1}^{\theta-1}\left(\mu_{t+1}h_{t+1}\right)^{1-\theta}-(1-\delta)\right]^{\frac{1}{\alpha}}-1 \\
&= \beta^{\frac{1}{\alpha}}\left[\theta\frac{y_{t+1}}{k_{t+1}}-(1-\delta)\right]^{\frac{1}{\alpha}}-1
\end{aligned}$$

and

$$\left(\frac{c_{t+1}}{c_t}\right)^{\alpha} = \beta\left[B-(1-\delta)\right]\left(\frac{k_{t+1}}{k_t}\right)^{\theta}\left(\frac{\mu_t h_t}{\mu_{t+1}h_{t+1}}\right)^{\theta}$$

Bringing together the simplified versions, we get

$$\gamma_t^k = \frac{y_t}{k_t} - \frac{c_t}{k_t} - \delta$$

$$\gamma_t^h = B\left(1 - \mu^*\right) - \delta.$$

$$\gamma_t^c = \beta^{\frac{1}{\alpha}} \left[\theta \frac{y}{k} - (1-\delta)\right]^{\frac{1}{\alpha}} - 1$$

$$\left(\frac{c_{t+1}}{c_t}\right)^{\alpha} = \beta \left[B - (1-\delta)\right] \left(\frac{k_{t+1}}{k_t}\right)^{\theta} \left(\frac{h_t}{h_{t+1}}\right)^{\theta}.$$

The last equation simplifies to

$$\gamma^* = \beta^{\frac{1}{\alpha}} \left[B - (1-\delta)\right]^{\frac{1}{\alpha}} - 1,$$

which can be calculated from the parameters of the model to give us the growth rates.. Using the other equation for consumption growth, we get

$$\frac{y}{k} = \frac{B}{\theta}.$$

To find the consumption capital ratio, we use

$$\gamma^* = \frac{y}{k} - \frac{c}{k} - \delta$$

and get

$$\frac{c}{k} = \frac{B}{\theta} - \gamma^* - \delta.$$

Finally, we find $\mu^*$ from

$$\mu^* = 1 - \frac{\gamma^* + \delta}{B}.$$