Crecimiento Economico
Class 9
Three topics

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.Today

- Three topics (related)
  - Solving an AK model with consumer maximization
  - Relating AK models with a model with Human capital
  - Solving a two sector model
    - Production function for goods (that are used for consumption and physical)
    - Production function for human capital

.CES sub-utility functions

- Constant elasticity of substitution sub-utility functions

$$u(c_t) = \frac{c_t^{1-\alpha} - 1}{1-\alpha}$$

- α is the inverse of the elasticy of intertemporal substitution

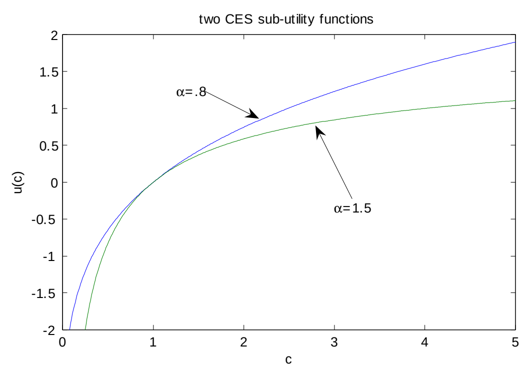
.CES sub-utility functions

- CES and log utility

$$\lim_{\alpha \to 1} \frac{c_t^{1-\alpha} - 1}{1-\alpha} = \ln c_t$$

Note: if we just put in $\alpha = 1$, we have

$$\frac{c_t^{1-\alpha} - 1}{1-\alpha} = \frac{c_t^{1-1} - 1}{1-1} = \frac{1-1}{0} = \frac{0}{0}$$



- l'Hopital's rule

$$\lim_{\alpha\to 1}\frac{c_t^{1-\alpha}-1}{1-\alpha}=\lim_{\alpha\to 1}\frac{\frac{d\left(c_t^{1-\alpha}-1\right)}{d\alpha}}{\frac{d(1-\alpha)}{d\alpha}}=\lim_{\alpha\to 1}\frac{-c_t^{1-\alpha}\ln c_t}{-1}=\ln c_t$$

First model: AK with optimization of consumers

- Households maximize

$$U_0=\sum_{t=0}^{\infty}\beta^t\frac{c_t^{1-\alpha}-1}{1-\alpha}$$

- with the budget constraints

$$k_{t+1}+c_t=y_t+(1-\delta)\,k_t$$

- and the production function

$$y_t=Ak_t$$

AK with optimization of consumers

- We will solve for constant growth paths (so we don't need the full policy function)
- Need first order conditions (necessary conditions)
- Use the Lagrangian

$$\mathcal{L}=\sum_{t=0}^{\infty}\beta^t\left[\frac{c_t^{1-\alpha}-1}{1-\alpha}+\lambda_t\left(k_{t+1}+c_t-Ak_t-(1-\delta)\,k_t\right)\right]$$

AK with optimization of consumers

- Fisrt order conditions are

$$\frac{\partial \mathcal{L}}{\partial c_s} = c_s^{-\alpha} + \lambda_s = 0$$
$$\frac{\partial \mathcal{L}}{\partial k_{s+1}} = \lambda_s - \lambda_{s+1}\beta\left[A + (1-\delta)\right] = 0$$

so

$$\left(\frac{c_{s+1}}{c_s}\right)^{\alpha} = \beta\left[A + (1-\delta)\right].$$

AK with optimization of consumers

- Growth rate of consumption is (from the first order conditions)

$$\gamma_t^c = \left(\frac{c_{s+1}}{c_s} - 1\right) = \beta^{\frac{1}{\alpha}}\left[A + (1-\delta)\right]^{\frac{1}{\alpha}} - 1,$$

- Growth rate of capital we find from

$$k_{t+1} - k_t = Ak_t - \delta k_t - c_t$$

and divide both sides by k_t to get

$$\gamma_t^k = \frac{k_{t+1} - k_t}{k_t} = A - \delta - \frac{c_t}{k_t}.$$

- Note that to have a constant growth rate of capital c_t/k_t needs to be constant

AK with optimization of consumers

- if c_t/k_t is a constant, then $\gamma_t^c = \gamma_t^k = \gamma_t^y = \gamma^*$
- Using this fact and combining the two growth rate equations, we get

$$\beta^{\frac{1}{\alpha}}\left[A + (1-\delta)\right]^{\frac{1}{\alpha}} - 1 = A - \delta - \frac{c}{k}$$

- In a stationary state growth path, c/k must be

$$\frac{c}{k} = A - \delta - \beta^{\frac{1}{\alpha}}\left[A + 1 - \delta\right]^{\frac{1}{\alpha}} + 1$$

- Everything on the right hand side are parameters, so we solve for c/k and then can find
$$\gamma^* = A - \delta - \frac{c}{k}.$$

AK with optimization of consumers (Example)

- Consider an economy with $A = 1$, $\alpha = 3$, $\delta = .1$, $\beta = .98$.
- Then
$$\frac{c}{k} = 1 - .1 - .98^{\frac{1}{3}} \left[1 + 1 - .1\right]^{\frac{1}{3}} + 1$$
- $c/k = 0.66975$
- and
$$\gamma^* = 1 - .1 - 0.66975 = 0.23025$$
- So the steady state growth rate is 23.025%

Finding the savings rate

- The budget constraint give us
$$s_t = y_t - c_t$$
and the saving rate (as a fraction of output) is
$$\frac{s_t}{y_t} = 1 - \frac{c_t}{y_t}.$$
- The production function is
$$y_t = Ak_t$$
we we can write
$$\widehat{s} = \frac{s_t}{y_t} = 1 - \frac{c_t}{y_t} = 1 - \frac{c_t}{Ak_t} = 1 - \frac{1}{1}0.66975 = 0.33025$$

AK with optimization of consumers

- Need to check if the utility function is well defined
- If
$$U_0 = \sum_{t=0}^{\infty} \beta^t \frac{c_t^{1-\alpha} - 1}{1 - \alpha} = \infty$$
we have a problem
- Need to find the conditions under which this has a finite value

AK with optimization of consumers

- Given a constant growth rate of consumption, $\gamma_t^c = \gamma^*$, we can write
$$\begin{aligned} c_t &= \left(1 + \gamma_t^c\right)^t c_0 \\ &= \beta^{\frac{t}{\alpha}} \left[A + (1 - \delta)\right]^{\frac{t}{\alpha}} c_0 \end{aligned}$$

- From

$$\sum_{t=0}^{\infty}\beta^t\frac{c_t^{1-\alpha}-1}{1-\alpha} = \sum_{t=0}^{\infty}\beta^t\frac{c_t^{1-\alpha}}{1-\alpha}-\frac{1}{1-\alpha}\sum_{t=0}^{\infty}\beta^t$$
$$= \frac{1}{1-\alpha}\sum_{t=0}^{\infty}\beta^t c_t^{1-\alpha}-\frac{1}{(1-\alpha)(1-\beta)}$$

- we need

$$\sum_{t=0}^{\infty}\beta^t c_t^{1-\alpha}<\infty$$

AK with optimization of consumers

- Substituting in our calculation for consumption, we have

$$\sum_{t=0}^{\infty}\beta^t\left(\beta^{\frac{t}{\alpha}}\left[A+(1-\delta)\right]^{\frac{t}{\alpha}}c_0\right)^{1-\alpha}=c_0^{1-\alpha}\sum_{t=0}^{\infty}\left(\beta\left(\beta\left[A+(1-\delta)\right]\right)^{\frac{(1-\alpha)}{\alpha}}\right)^t$$

- and we need

$$\beta\left(\beta\left[A+(1-\delta)\right]\right)^{\frac{(1-\alpha)}{\alpha}}<1$$

- which occurs when

$$\left[A+(1-\delta)\right]^{(1-\alpha)}<\beta^{-1}$$

- Note that for our example we are using $\alpha=3$, so the condition for the example is

$$\frac{1}{\left[A+(1-\delta)\right]^2}<\frac{1}{\beta}$$

AK with optimization of consumers

- Result:
- We can find an equilibrium with optimizing consumers in a model with AK production function
- We find the constant growth rate path
- There, all variables grow at the same rate
- Need to check that the transversality condition holds and therefore that utility is difined

Second model: AK models as models of human capital

- We can consider the conditions under which a model with production that uses human capital can become an AK type model

- Consider the production function

$$Y_t = BK_t^{\theta} H_t^{1-\theta}$$

- and assume that both one unit of capital and one unit of human capital can be produced with one unit of the good, so the budget constraint can be written as (we assume that depreciation is the same for both types of capital)

$$K_{t+1} + H_{t+1} = w_t H_t + r_t K_t + (1-\delta) H_t + (1-\delta) K_t.$$

AK models as models of human capital

- The return on a unit of human capital is $w_t - \delta$
- The return on a unit of physical capital is $r_t - \delta$
- In equilibrium, these returns must be equal (since the cost of a unit of each is the same)
- From competitive factor markets, we have

$$w_t = (1-\theta) BK_t^{\theta} H_t^{-\theta} = (1-\theta) \frac{Y_t}{H_t}$$

and

$$r_t = \theta BK_t^{\theta-1} H_t^{1-\theta} = \theta \frac{Y_t}{K_t}.$$

AK models as models of human capital

- Since depreciation is the same for both physical and human capital, the equilibrium condition is

$$w_t = r_t$$

- This can be written as

$$(1-\theta) \frac{Y_t}{H_t} = \theta \frac{Y_t}{K_t}$$

- which can be simplified to get

$$H_t = \frac{(1-\theta)}{\theta} K_t.$$

AK models as models of human capital

- Putting this last result into the production function, we get that in equilibrium

$$\begin{aligned} Y_t &= BK_t^\theta H_t^{1-\theta} = BK_t^\theta \left(\frac{1-\theta}{\theta} K_t \right)^{1-\theta} \\ &= B \left(\frac{1-\theta}{\theta} \right)^{1-\theta} K_t \\ &= AK_t \end{aligned}$$

where the constant A is simply

$$A = B \left(\frac{1-\theta}{\theta} \right)^{1-\theta}.$$

AK models as models of human capital

- Result:
- with same depreciation rates and same cost of production,
- a model of production with production function

$$Y_t = BK_t^\theta H_t^{1-\theta}$$

- becomes a model with production

$$Y_t = AK_t$$

- where

$$A = B \left(\frac{1-\theta}{\theta} \right)^{1-\theta}.$$

Third Model: Two sector model with human capital (Uzawa and Lucas)

- Idea: two production sectors
- One sector produces the one good
  - Good used for consumption and for physical capital
  - Production uses physical capital and human capital
  - No direct labor (nor in utility function)
- Other sector produces human capital (education sector)
  - Is used for producing goods and human capital
  - Using human capital in this sector reduces amount used in production of goods

– Idea is that with more human capital it is easier to learn

.Two sector model with human capital

- General version first (we simplify in a bit)
- Budget constraint for goods production (with production function included) is

$$K_{t+1} = AK_t(y)^{\theta} H_t(y)^{1-\theta} - C_t + (1 - \delta_k) K_t$$

- Budget constraint for human capital production is

$$H_{t+1} = BK_t(h)^{\eta} H_t(h)^{1-\eta} + (1 - \delta_h) H_t.$$

- The economy has feasibility constraints of

$$K_t = K_t(y) + K_t(h)$$

and

$$H_t = H_t(y) + H_t(h).$$

.Two sector model with human capital

- Let μ_t be the fraction of human capital used in production of goods
- then $1 - \mu_t$ is fraction used in production of more human capital
- Get equations

$$H_t(y) = \mu_t H_t$$

and

$$H_t(h) = (1 - \mu_t) H_t.$$

- This will be a useful definition

.Two sector model with human capital

- We assume that the goods sector is more intensive in physical capital
- this implies that

$$\theta > \eta.$$

- Uzawa and Lucas (and we) use the limit condition

$$\eta = 0$$

- Education uses only human capital
- Goods production uses both human and physical capital

.Two sector model with human capital

- Putting this assumption into model give

$$K_{t+1} = AK_t^{\theta} \left(\mu_t H_t\right)^{1-\theta} - C_t + \left(1 - \delta_k\right) K_t$$

and

$$H_{t+1} = B \left(1 - \mu_t\right) H_t + \left(1 - \delta_h\right) H_t.$$

- Written in per capita terms (here we will not worry about population growth but are keeping in the spirit of the papers)

$$k_{t+1} = Ak_t^{\theta} \left(\mu_t h_t\right)^{1-\theta} - c_t + \left(1 - \delta_k\right) k_t$$

and

$$h_{t+1} = B \left(1 - \mu_t\right) h_t + \left(1 - \delta_h\right) h_t.$$

.Two sector model with human capital

- Maximization problem of the households is: subject to the above budget constraints max

$$U_0 = \sum_{t=0}^{\infty} \beta^t \frac{c_t^{1-\alpha} - 1}{1 - \alpha}$$

- Same utility function as in the first model
- Lagrangian (we can use this because we will only look at constant growth rate paths) is

$$\begin{aligned} \mathcal{L} &= \sum_{t=0}^{\infty} \beta^t \left[\frac{c_t^{1-\alpha} - 1}{1 - \alpha} \right. \\ &\quad + \lambda_t^1 \left(k_{t+1} - Ak_t^{\theta} \left(\mu_t h_t\right)^{1-\theta} + c_t - \left(1 - \delta_k\right) k_t \right) \\ &\quad \left. + \lambda_t^2 \left(h_{t+1} - B \left(1 - \mu_t\right) h_t - \left(1 - \delta_h\right) h_t \right) \right]. \end{aligned}$$

.Two sector model with human capital

- First order conditions are (wrt: c_t, k_{t+1}, h_{t+1}, u_t)

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial c_t} &= c_t^{-\alpha} + \lambda_t^1 = 0 \\ \frac{\partial \mathcal{L}}{\partial k_{t+1}} &= \lambda_t^1 - \beta \lambda_{t+1}^1 \left[\theta A k_{t+1}^{\theta-1} \left(\mu_{t+1} h_{t+1}\right)^{1-\theta} - \left(1 - \delta_k\right) \right] = 0 \\ \frac{\partial \mathcal{L}}{\partial h_{t+1}} &= 0 = -\beta \lambda_{t+1}^1 \left(1 - \theta\right) \mu_{t+1} A k_{t+1}^{\theta} \left(\mu_{t+1} h_{t+1}\right)^{-\theta} \\ &\quad + \lambda_t^2 - \beta \lambda_{t+1}^2 \left[B \left(1 - \mu_{t+1}\right) - \left(1 - \delta_h\right) \right] \\ \frac{\partial \mathcal{L}}{\partial \mu_t} &= -\lambda_t^1 \left(1 - \theta\right) h_t A k_t^{\theta} \left(\mu_t h_t\right)^{-\theta} + \lambda_t^2 B h_t = 0 \end{aligned}$$

- How to solve

.Two sector model with human capital

- Simplified first order conditions (after removing the multipliers)

$$\begin{aligned}\left(\frac{c_{t+1}}{c_t}\right)^{\alpha} &= \beta\left[\theta A k_{t+1}^{\theta-1}\left(\mu_{t+1}h_{t+1}\right)^{1-\theta}-(1-\delta)\right] \\ \frac{A k_t^{\theta}\left(\mu_t h_t\right)^{-\theta}}{c_t^{\alpha}} &= \beta\left[B-(1-\delta)\right]\frac{A k_{t+1}^{\theta}\left(\mu_{t+1}h_{t+1}\right)^{-\theta}}{c_{t+1}^{\alpha}}\end{aligned}$$

- and the two budget constraints

$$k_{t+1} = A k_t^{\theta}\left(\mu_t h_t\right)^{1-\theta} - c_t + (1-\delta)\,k_t$$

and

$$h_{t+1} = B\left(1-\mu_t\right)h_t + (1-\delta)\,h_t.$$

.Two sector model with human capital

- We solve for constant growth rates: $\gamma_t^k = \gamma_t^h = \gamma_t^y = \gamma_t^c = \gamma^*$
- all variables need to grow at the same rate
- At constant growth μ_t is a constant so $\mu_t = \mu^*$
- Ratios between variables will also be constant: c/k, y/k, for example

.Two sector model with human capital

- The growth rate equations are

$$\begin{aligned}\gamma_t^k &= \frac{k_{t+1}}{k_t}-1 = A k_t^{\theta-1}\left(\mu_t h_t\right)^{1-\theta} - \frac{c_t}{k_t} - \delta \\ &= \frac{y_t}{k_t} - \frac{c_t}{k_t} - \delta\end{aligned}$$

$$\gamma_t^h = \frac{h_{t+1}}{h_t} - 1 = B\left(1-\mu_t\right) - \delta.$$

$$\begin{aligned}\gamma_t^c &= \left(\frac{c_{t+1}}{c_t}-1\right) = \beta^{\frac{1}{\alpha}}\left[\theta A k_{t+1}^{\theta-1}\left(\mu_{t+1}h_{t+1}\right)^{1-\theta}-(1-\delta)\right]^{\frac{1}{\alpha}} - 1 \\ &= \beta^{\frac{1}{\alpha}}\left[\theta\frac{y_{t+1}}{k_{t+1}}-(1-\delta)\right]^{\frac{1}{\alpha}} - 1\end{aligned}$$

$$\left(\frac{c_{t+1}}{c_t}\right)^{\alpha} = \beta\left[B-(1-\delta)\right]\left(\frac{k_{t+1}}{k_t}\right)^{\theta}\left(\frac{\mu_t h_t}{\mu_{t+1}h_{t+1}}\right)^{\theta}$$

.Two sector model with human capital

- The growth rate equations simplify to

$$\gamma_t^k = \frac{y_t}{k_t} - \frac{c_t}{k_t} - \delta$$

$$\gamma_t^h = B\left(1-\mu^*\right) - \delta.$$

$$\gamma_t^c = \beta^{\frac{1}{\alpha}}\left[\theta\frac{y}{k} - (1-\delta)\right]^{\frac{1}{\alpha}} - 1$$

$$\left(\frac{c_{t+1}}{c_t}\right)^{\alpha} = \beta\left[B-(1-\delta)\right]\left(\frac{k_{t+1}}{k_t}\right)^{\theta}\left(\frac{h_t}{h_{t+1}}\right)^{\theta}.$$

- The last equation simplifies to (the growth rate equation for the economy)

$$\gamma^* = \beta^{\frac{1}{\alpha}}\left[B-(1-\delta)\right]^{\frac{1}{\alpha}} - 1,$$

.Two sector model with human capital

- Using

$$\gamma_t^c = \beta^{\frac{1}{\alpha}}\left[\theta\frac{y}{k} - (1-\delta)\right]^{\frac{1}{\alpha}} - 1$$

  and

$$\gamma^* = \beta^{\frac{1}{\alpha}}\left[B-(1-\delta)\right]^{\frac{1}{\alpha}} - 1,$$

- we get

$$\beta^{\frac{1}{\alpha}}\left[\theta\frac{y}{k} - (1-\delta)\right]^{\frac{1}{\alpha}} - 1 = \beta^{\frac{1}{\alpha}}\left[B-(1-\delta)\right]^{\frac{1}{\alpha}} - 1$$

- or

$$\frac{y}{k} = \frac{B}{\theta}.$$

.Two sector model with human capital

- The rest of the model follows naturally
- we use

$$\gamma^* = \frac{y}{k} - \frac{c}{k} - \delta$$

  to get

$$\frac{c}{k} = \frac{B}{\theta} - \gamma^* - \delta.$$

- Finally, we get μ^* from

$$\mu^* = 1 - \frac{\gamma^* + \delta}{B}.$$

.Two sector model with human capital

- Result:
- Relatively easy to find constant growth paths for two sector model
- How would one find the policy function for non-constant growth paths?
- What would you need to do?
- These you look for as approximations