

# Economic Growth

## Class 8

### More models of the AK type

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Our three models type AK

- The AK model:  $Y_t = AK_t$
- The Romer model with externality of aggregate per worker capital:
  - Production of firm  $i$  is equal to  $Y(i)_t = AK(i)_t^\theta H(i)_t^{1-\theta} \kappa_t^\eta$
  - where  $\kappa = K_t/H_t$
  - and where  $\theta + \eta = 1$
- Government production model
  - production function is  $Y_t = K_t^\theta G_t^{1-\theta}$
  - and the government budget constraint is  $\tau Y_t = G_t$
- All of these function like AK models
  - All fail to have decreasing marginal product to capital
  - All fail the Inada conditions

Today: variations on AK models

- Sobelow model: combining Solow and AK models
- Harrod-Domar Model
- Solving an AK model with consumer maximization
- Relating AK models with a model with Human capital

The Sobelow model: Combining Solow and AK

- Assume that the production function is

$$Y_t = AK_t + BK_t^\theta H_t^{1-\theta}$$

- This model gives constant returns to scale

$$A\lambda K_t + B(\lambda K_t)^\theta (\lambda H_t)^{1-\theta} = A\lambda K_t + \lambda BK_t^\theta H_t^{1-\theta} = \lambda Y_t$$

- Returns to capital and labor are positive

$$\frac{\partial Y_t}{\partial K_t} = A + \theta BK_t^{\theta-1} H_t^{1-\theta} > 0$$

$$\frac{\partial Y_t}{\partial H_t} = (1 - \theta) BK_t^\theta H_t^{-\theta} > 0$$

- and are decreasing (check out the second derivatives)

The Sobelow model: Combining Solow and AK

- The model does not fulfill the Inada conditions (one of them)

$$\lim_{K \rightarrow \infty} \frac{\partial Y_t}{\partial K_t} = A \neq 0$$

$$\lim_{K \rightarrow 0} \frac{\partial Y_t}{\partial K_t} = \infty$$

$$\lim_{H \rightarrow \infty} \frac{\partial Y_t}{\partial H_t} = 0$$

$$\lim_{H \rightarrow 0} \frac{\partial Y_t}{\partial H_t} = \infty$$

- Fails in one of the conditions

The Sobelow model: Combining Solow and AK

- Find the per worker production function

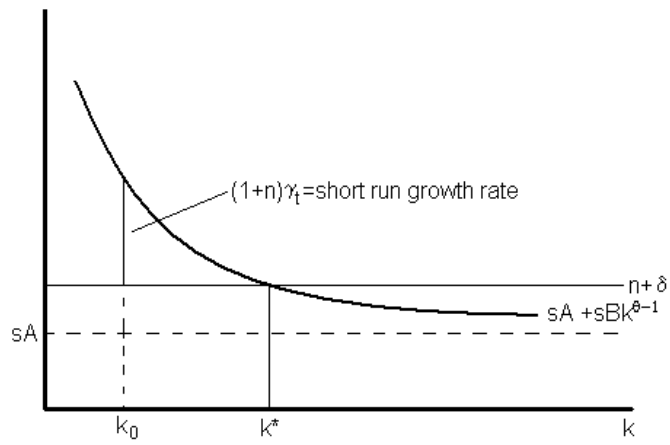
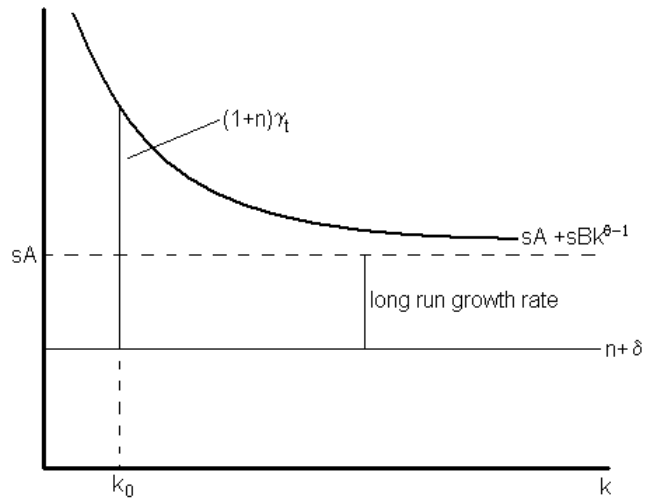
$$y_t = Ak_t + Bk_t^\theta$$

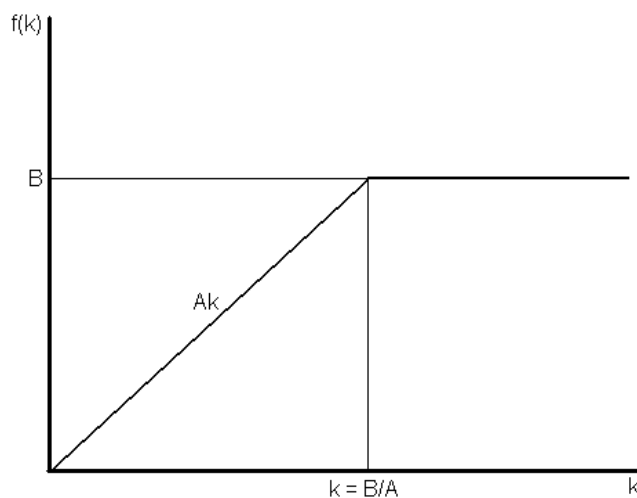
- Use the law of motion of capital with the conditions for savings added

$$\begin{aligned} (1+n)k_{t+1} &= (1-\delta)k_t + sf(k_t) \\ &= (1-\delta)k_t + sAk_t + sBk_t^\theta \end{aligned}$$

- Write as the growth rate of capital

$$(1+n)\gamma_t = (1+n) \frac{(k_{t+1} - k_t)}{k_t} = sA + sBk_t^{\theta-1} - (n+\delta)$$





The Sobelow model: Combining Solow and AK  
The Sobelow model: Combining Solow and AK  
Harrod-Domar Model

- Very early growth model
  - Harrod (1939), Domar (1946)

- Based on Leontief production functions

$$Y_t = \min [AK_t, BH_t]$$

- which in per worker terms is

$$y_t = \min [Ak_t, B]$$

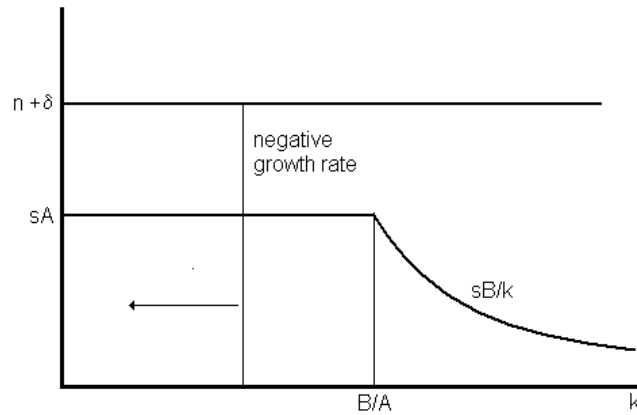
- this says that
  - if  $k_t$  is small enough so that  $Ak_t < B$ , then  $y_t = Ak_t$
  - if  $k_t$  is big enough so that  $Ak_t \geq B$ , then  $y_t = B$

- The dividing point is when  $k_t = B/A$

Harrod-Domar Model (the production function)  
Harrod-Domar Model

- The usual law of motion of capital is

$$(1 + n) k_{t+1} = (1 - \delta) k_t + s f(k_t)$$



- This now needs to be written as

$$(1+n)(k_{t+1} - k_t) = \begin{cases} sAk_t - (n+\delta)k_t & \text{if } k < B/A \\ sB - (n+\delta)k_t & \text{if } k \geq B/A \end{cases}$$

- or as

$$(1+n)\gamma_t = \begin{cases} sA - (n+\delta) & \text{if } k < B/A \\ sB/k_t - (n+\delta) & \text{if } k \geq B/A \end{cases}$$

Harrod-Domar Model (three possible equilibria: 1)

- equilibrium is at  $k = 0$

Harrod-Domar Model (three possible equilibria: 2)

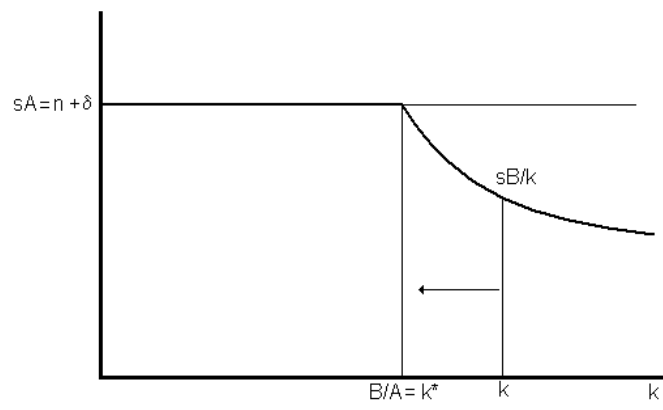
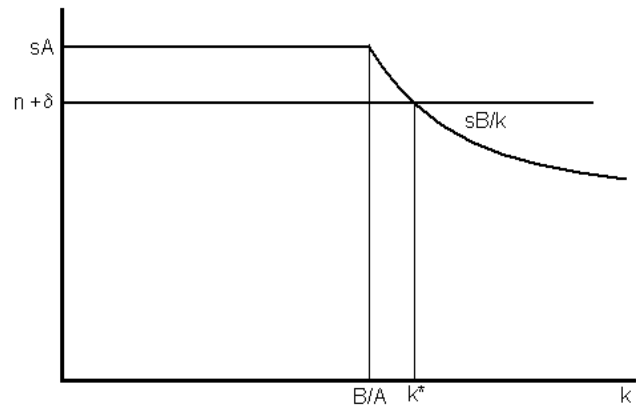
- stationary state at  $k^*$  but not all capital is utilized

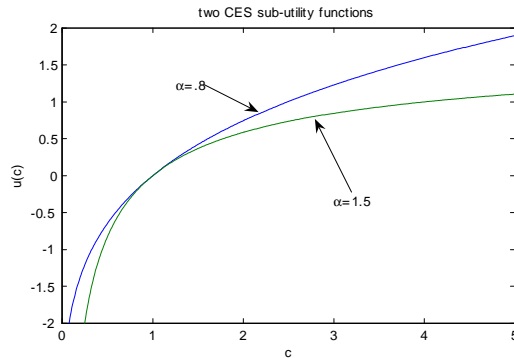
Harrod-Domar Model (three possible equilibria: 3)

- Knife edge equilibrium

Harrod-Domar Model (possibility 3)

- If initial  $k$  is above  $B/A$ , the economy goes to  $k^* = B/A$
- if initial  $k$  is below  $B/A$ , the economy stays there (no growth)
- Harrod and others at the University of Cambridge argued
  - the savings rate is not constant
  - with higher growth, the savings rate grows
  - workers have different savings rate = marginal propensity to save than capitalists





- as economy grows, more income goes to the workers and the savings rate declines
- think about what this means

Optimizing AK: CES sub-utility functions

- Constant elasticity of substitution sub-utility functions

$$u(c_t) = \frac{c_t^{1-\alpha} - 1}{1-\alpha}$$

- $\alpha$  is the inverse of the elasticity of intertemporal substitution

CES sub-utility functions

- CES and log utility: fact

$$\lim_{\alpha \rightarrow 1} \frac{c_t^{1-\alpha} - 1}{1-\alpha} = \ln c_t$$

- Note: if we just put in  $\alpha = 1$ , we have

$$\frac{c_t^{1-\alpha} - 1}{1-\alpha} = \frac{c_t^{1-1} - 1}{1-1} = \frac{1-1}{0} = \frac{0}{0}$$

- need to use l'Hopital's rule

$$\lim_{\alpha \rightarrow 1} \frac{c_t^{1-\alpha} - 1}{1-\alpha} = \lim_{\alpha \rightarrow 1} \frac{\frac{d(c_t^{1-\alpha} - 1)}{d\alpha}}{\frac{d(1-\alpha)}{d\alpha}} = \lim_{\alpha \rightarrow 1} \frac{-c_t^{1-\alpha} \ln c_t}{-1} = \ln c_t$$

First model: AK with optimization of consumers

- Households maximize

$$U_0 = \sum_{t=0}^{\infty} \beta^t \frac{c_t^{1-\alpha} - 1}{1-\alpha}$$

- with the budget constraints

$$k_{t+1} + c_t = y_t + (1 - \delta) k_t$$

- and the production function

$$y_t = Ak_t$$

AK with optimization of consumers

- We will solve for constant growth paths (so we don't need the full policy function)
- Need first order conditions (necessary conditions)
- Use the Lagrangian

$$\mathcal{L} = \sum_{t=0}^{\infty} \beta^t \left[ \frac{c_t^{1-\alpha} - 1}{1-\alpha} + \lambda_t (k_{t+1} + c_t - Ak_t - (1 - \delta) k_t) \right]$$

AK with optimization of consumers

- First order conditions are

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial c_s} &= c_s^{-\alpha} + \lambda_s = 0 \\ \frac{\partial \mathcal{L}}{\partial k_{s+1}} &= \lambda_s - \lambda_{s+1} \beta [A + (1 - \delta)] = 0 \end{aligned}$$

so

$$\left( \frac{c_{s+1}}{c_s} \right)^\alpha = \beta [A + (1 - \delta)].$$

AK with optimization of consumers

- Growth rate of consumption is (from the first order conditions)

$$\gamma_t^c = \left( \frac{c_{s+1}}{c_s} - 1 \right) = \beta^{\frac{1}{\alpha}} [A + (1 - \delta)]^{\frac{1}{\alpha}} - 1,$$

- Growth rate of capital we find from

$$k_{t+1} - k_t = Ak_t - \delta k_t - c_t$$

and divide both sides by  $k_t$  to get

$$\gamma_t^k = \frac{k_{t+1} - k_t}{k_t} = A - \delta - \frac{c_t}{k_t}.$$

- Note that to have a constant growth rate of capital  $c_t/k_t$  needs to be constant

AK with optimization of consumers

- if  $c_t/k_t$  is a constant, then  $\gamma_t^c = \gamma_t^k = \gamma_t^y = \gamma^*$
- Using this fact and combining the two growth rate equations, we get

$$\beta^{\frac{1}{\alpha}} [A + (1 - \delta)]^{\frac{1}{\alpha}} - 1 = A - \delta - \frac{c}{k}$$

- In a stationary state growth path,  $c/k$  must be

$$\frac{c}{k} = A - \delta - \beta^{\frac{1}{\alpha}} [A + 1 - \delta]^{\frac{1}{\alpha}} + 1$$

- Everything on the right hand side are parameters, so we solve for  $c/k$  and then can find

$$\gamma^* = A - \delta - \frac{c}{k}.$$

AK with optimization of consumers (Example)

- Consider an economy with  $A = 1$ ,  $\alpha = 3$ ,  $\delta = .1$ ,  $\beta = .98$ .

- Then

$$\frac{c}{k} = 1 - .1 - .98^{\frac{1}{3}} [1 + 1 - .1]^{\frac{1}{3}} + 1$$

- $c/k = 0.66975$

- and

$$\gamma^* = 1 - .1 - 0.66975 = 0.23025$$

- So the steady state growth rate is 23.025%

Finding the savings rate

- The budget constraint give us

$$s_t = y_t - c_t$$

and the saving rate (as a fraction of output) is

$$\frac{s_t}{y_t} = 1 - \frac{c_t}{y_t}.$$

- The production function is

$$y_t = Ak_t$$

we we can write

$$\hat{s} = \frac{s_t}{y_t} = 1 - \frac{c_t}{y_t} = 1 - \frac{c_t}{Ak_t} = 1 - \frac{1}{1} 0.66975 = 0.33025$$

AK with optimization of consumers

- Need to check if the utility function is well defined
- If

$$U_0 = \sum_{t=0}^{\infty} \beta^t \frac{c_t^{1-\alpha} - 1}{1-\alpha} = \infty$$

we have a problem

- Need to find the conditions under which this has a finite value

AK with optimization of consumers

- Given a constant growth rate of consumption,  $\gamma_t^c = \gamma^*$ , we can write

$$\begin{aligned} c_t &= (1 + \gamma_t^c)^t c_0 \\ &= \beta^{\frac{t}{\alpha}} [A + (1 - \delta)]^{\frac{t}{\alpha}} c_0 \end{aligned}$$

- From

$$\begin{aligned} \sum_{t=0}^{\infty} \beta^t \frac{c_t^{1-\alpha} - 1}{1-\alpha} &= \sum_{t=0}^{\infty} \beta^t \frac{c_t^{1-\alpha}}{1-\alpha} - \frac{1}{1-\alpha} \sum_{t=0}^{\infty} \beta^t \\ &= \frac{1}{1-\alpha} \sum_{t=0}^{\infty} \beta^t c_t^{1-\alpha} - \frac{1}{(1-\alpha)(1-\beta)} \end{aligned}$$

- we need

$$\sum_{t=0}^{\infty} \beta^t c_t^{1-\alpha} < \infty$$

AK with optimization of consumers

- Substituting in our calculation for consumption, we have

$$\sum_{t=0}^{\infty} \beta^t \left( \beta^{\frac{t}{\alpha}} [A + (1 - \delta)]^{\frac{t}{\alpha}} c_0 \right)^{1-\alpha} = c_0^{1-\alpha} \sum_{t=0}^{\infty} \left( \beta (\beta [A + (1 - \delta)])^{\frac{(1-\alpha)}{\alpha}} \right)^t$$

- and we need

$$\beta (\beta [A + (1 - \delta)])^{\frac{(1-\alpha)}{\alpha}} < 1$$

- which occurs when

$$[A + (1 - \delta)]^{(1-\alpha)} < \beta^{-1}$$

- Note that for our example we are using  $\alpha = 3$ , so the condition for the example is

$$\frac{1}{[A + (1 - \delta)]^2} < \frac{1}{\beta}$$

AK with optimization of consumers

- Result:
- We can find an equilibrium with optimizing consumers in a model with AK production function
- We find the constant growth rate path
- There, all variables grow at the same rate
- Need to check that the transversality condition holds and therefore that utility is defined

Second model: AK models as models of human capital

- We can consider the conditions under which a model with production that uses human capital can become an AK type model
- Consider the production function

$$Y_t = BK_t^\theta H_t^{1-\theta}$$

- and assume that both one unit of capital and one unit of human capital can be produced with one unit of the good, so the budget constraint can be written as (we assume that depreciation is the same for both types of capital)

$$K_{t+1} + H_{t+1} = w_t H_t + r_t K_t + (1 - \delta) H_t + (1 - \delta) K_t.$$

AK models as models of human capital

- The return on a unit of human capital is  $w_t - \delta$
- The return on a unit of physical capital is  $r_t - \delta$
- In equilibrium, these returns must be equal (since the cost of a unit of each is the same)
- From competitive factor markets, we have

$$w_t = (1 - \theta) BK_t^\theta H_t^{-\theta} = (1 - \theta) \frac{Y_t}{H_t}$$

and

$$r_t = \theta BK_t^{\theta-1} H_t^{1-\theta} = \theta \frac{Y_t}{K_t}.$$

AK models as models of human capital

- Since depreciation is the same for both physical and human capital, the equilibrium condition is

$$w_t = r_t$$

- This can be written as

$$(1 - \theta) \frac{Y_t}{H_t} = \theta \frac{Y_t}{K_t}$$

- which can be simplified to get

$$H_t = \frac{(1 - \theta)}{\theta} K_t.$$

AK models as models of human capital

- Putting this last result into the production function, we get that in equilibrium

$$\begin{aligned} Y_t &= BK_t^\theta H_t^{1-\theta} = BK_t^\theta \left( \frac{1-\theta}{\theta} K_t \right)^{1-\theta} \\ &= B \left( \frac{1-\theta}{\theta} \right)^{1-\theta} K_t \\ &= AK_t \end{aligned}$$

where the constant  $A$  is simply

$$A = B \left( \frac{1-\theta}{\theta} \right)^{1-\theta}.$$

AK models as models of human capital

- Result:
- with same depreciation rates and same cost of production,
- a model of production with production function

$$Y_t = BK_t^\theta H_t^{1-\theta}$$

- becomes a model with production

$$Y_t = AK_t$$

- where

$$A = B \left( \frac{1-\theta}{\theta} \right)^{1-\theta}.$$