

Economic Growth

Class 7

Endogenous growth

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Endogenous growth: the idea

- We want a model where economies can keep growing
- Where growth does not need to depend on exogenous technology growth
 - So it is not about moving the stationary state
 - It is about economies that don't have stationary states
- The models have AK as their underlying basis

Paul Romer's model on externalities on capital

- Assume that there are some externalities from producing
- These externalities can come from
 - my capital is more productive based on total capital (or average capital per worker)
 - my capital is more productive if I produced more last period (learning by doing)
 - my capital is more productive if everyone knows more (knowledge spillovers)
- Example of a production function for the firm j with these characteristics

$$Y_t^j = A \left(K_t^j\right)^\theta \left(H_t^j\right)^{1-\theta} \kappa_t^\eta$$

where κ_t is the externality

Romer's version

- let $\kappa_t = k_t = K_t/H_t$
- the above production function is then

$$\begin{aligned} Y_t^j &= A \left(K_t^j \right)^\theta \left(H_t^j \right)^{1-\theta} \kappa_t^\eta \\ &= A \left(K_t^j \right)^\theta \left(H_t^j \right)^{1-\theta} k_t^\eta \\ &= A \left(K_t^j \right)^\theta \left(H_t^j \right)^{1-\theta} \left(\frac{K_t}{H_t} \right)^\eta \end{aligned}$$

- This can be written as

$$Y_t^j = A \left(\frac{K_t^j}{H_t^j} \right)^\theta H_t^j \left(\frac{K_t}{H_t} \right)^\eta$$

Romer's version

- Each firm maximizes its profits subject to its costs and production function
- This problem is to max

$$\pi_t^j = Y_t^j - w_t H_t^j - r_t K_t^j$$

- or

$$\pi_t^j = A \left(\frac{K_t^j}{H_t^j} \right)^\theta H_t^j \left(\frac{K_t}{H_t} \right)^\eta - w_t H_t^j - r_t K_t^j$$

- Notice that the firm only controls K_t^j and H_t^j

Romer's version

- The first order conditions are

$$w_t = (1 - \theta) A \left(\frac{K_t^j}{H_t^j} \right)^\theta \left(\frac{K_t}{H_t} \right)^\eta$$

and

$$r_t = \theta A \left(\frac{K_t^j}{H_t^j} \right)^{\theta-1} \left(\frac{K_t}{H_t} \right)^\eta$$

- Since for all firms, $\theta, A, K_t, H_t,$ and η are given, all choose the same K_t^j/H_t^j and this will end up equal to K_t/H_t

Romer's version

- The production function for firm j can be written as

$$Y_t^j = A \left(\frac{K_t}{H_t} \right)^{\theta+\eta} H_t^j$$

- summing across the J firms we get

$$Y_t = \sum_{j=1}^J Y_t^j = \sum_{j=1}^J A \left(\frac{K_t}{H_t} \right)^{\theta+\eta} H_t^j = A \left(\frac{K_t}{H_t} \right)^{\theta+\eta} \sum_{j=1}^J H_t^j = A \left(\frac{K_t}{H_t} \right)^{\theta+\eta} H_t$$

- or the aggregate production function

$$Y_t = AK_t^{\theta+\eta} H_t^{1-\theta-\eta}$$

- Of special interest is when $\theta + \eta = 1$ or $\theta + \eta > 1$

Romer's version

- Write the aggregate production function in per worker terms

$$Y_t = AK_t^{\theta+\eta} H_t^{1-\theta-\eta}$$

gives

$$y_t = \frac{Y_t}{H_t} = \frac{AK_t^{\theta+\eta} H_t^{1-\theta-\eta}}{H_t} = AK_t^{\theta+\eta} H_t^{-\theta-\eta} = Ak_t^{\theta+\eta}$$

or

$$y_t = Ak_t^{\theta+\eta}$$

Romer's version

- Assume a Solow type model where

$$\begin{aligned} (1+n)k_{t+1} &= (1-\delta)k_t + i_t \\ s_t &= i_t \\ s_t &= sy_t \end{aligned}$$

and the production function

$$y_t = Ak_t^{\theta+\eta}$$

- These can be combined to get the function

$$(1+n)k_{t+1} = (1-\delta)k_t + sAk_t^{\theta+\eta}$$

- The growth rate of capital per capita in period t , γ_t , is calculated as

$$\begin{aligned} \gamma_t &= \frac{k_{t+1}}{k_t} - 1 = \frac{(1-\delta)}{(1+n)} - 1 + \frac{sA}{(1+n)} k_t^{\theta+\eta-1} \\ &= -\frac{\delta+n}{(1+n)} + \frac{sA}{(1+n)} k_t^{\theta+\eta-1} \end{aligned}$$

.What happens when

- $\theta + \eta < 1$
- The growth rate can be written as

$$\gamma_t = \frac{sA}{(1+n)k_t^{1-\theta-\eta}} - \frac{(\delta+n)}{(1+n)}$$

where $1 - \theta - \eta > 0$ and

$$\frac{d\gamma_t}{dk_t} = (\theta + \eta - 1) \frac{sA}{(1+n)k_t^{2-\theta-\eta}} < 0$$

- when $\theta + \eta < 1$ the rate of growth declines with capital
- there is a stationary state when

$$(n + \delta) \bar{k} = sA\bar{k}^{-\theta+\eta}$$

or when

$$\bar{k} = \left(\frac{sA}{n+\delta} \right)^{\frac{1}{1-\theta-\eta}}$$

.What happens when

- $\theta + \eta < 1$
- There are two stationary states: $\bar{k} = 0$ and $\bar{k} = \left(\frac{sA}{n+\delta} \right)^{\frac{1}{1-\theta-\eta}}$
- The model behaves like the standard Solow model between these points
 - grows toward the upper stationary state
 - location of upper stationary state depends on country's value for s, A, n, δ, θ and η

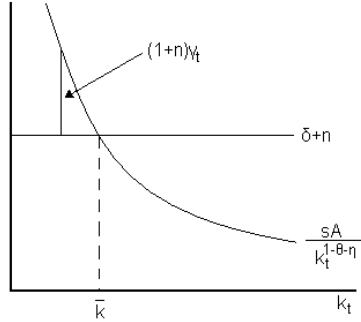
.What happens when

- $\theta + \eta < 1$
- The growth rate equation is

$$(1+n)\gamma_t = \frac{sA}{k_t^{1-\theta-\eta}} - (\delta+n)$$

- This gives the graph

.What happens when



- $\theta + \eta = 1$
- From above calculations, growth is

$$\gamma_t = \frac{sA}{(1+n)k_t^{1-\theta-\eta}} - \frac{(\delta+n)}{(1+n)}$$

- but in this case, $1 - \theta - \eta = 0$,
- so growth is simply

$$(1+n)\gamma_t = sA - (\delta+n)$$

- The result of this model is the same as an AK model

What happens when

- $\theta + \eta = 1$
- There is no stationary state
- growth rate is constant at all capital stock levels and positive if $sA > (\delta+n)$

What happens when

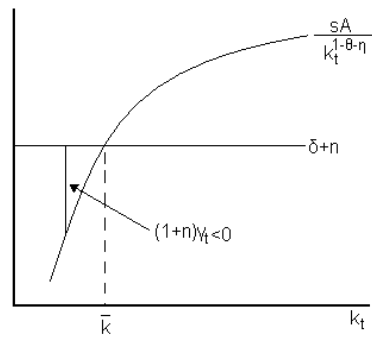
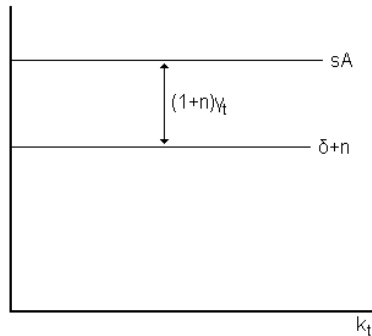
- $\theta + \eta > 1$
- From above calculations, growth is

$$\gamma_t = \frac{sA}{(1+n)k_t^{1-\theta-\eta}} - \frac{(\delta+n)}{(1+n)}$$

- but $1 - \theta - \eta < 0$, so the growth rate can be written as

$$\gamma_t = \frac{sAk_t^{\theta+\eta-1}}{(1+n)} - \frac{(\delta+n)}{(1+n)}$$

where $\theta + \eta - 1 > 0$



- This means that the growth rate grows with the capital stock

What happens when

- $\theta + \eta > 1$
- the stationary state is a repeller
- economies below \bar{k} shrink and those above \bar{k} grow even faster
- This is NOT observed in the data

Characteristics of $\theta + \mu = 1$ version

- Growth rate is the same for poor and rich countries
 - poor countries have less starting capital
- wage and rental rates in the economy are

$$w_t = (1 - \theta) A \left(\frac{K_t^j}{H_t^j} \right)^\theta \left(\frac{K_t}{H_t} \right)^\eta = (1 - \theta) A \left(\frac{K_t}{H_t} \right)^{\eta + \theta} = (1 - \theta) A \frac{K_t}{H_t}$$

and

$$r_t = \theta A \left(\frac{K_t^j}{H_t^j} \right)^{\theta-1} \left(\frac{K_t}{H_t} \right)^\eta = \theta A \left(\frac{K_t}{H_t} \right)^{\eta+\theta-1} = \theta A$$

- Rental rates are the same everywhere but rich countries have higher wages

An alternative formulation

- Suppose that instead of $\kappa = k$, the per worker capital, the externality was based on $\kappa = K$, the stock of capital
- To keep things simple, assume that population is constant: $n = 0$
- Write the production function as

$$\begin{aligned} Y_t &= AK_t^\theta H_t^{1-\theta} \kappa_t^\eta \\ &= AK_t^\theta H_t^{1-\theta} K_t^\eta \\ &= AK_t^{\theta+\eta} H_t^{1-\theta} \end{aligned}$$

- Dividing by H_t , we get

$$y_t = Ak_t^\theta K_t^\eta$$

- K_t^η can be written as $K_t^\eta = k_t^\eta H_t^\eta$
- the per worker production function can be written as

$$y_t = Ak_t^{\theta+\eta} H_t^\eta$$

An alternative formulation

- Use the rest of the model

$$\begin{aligned} k_{t+1} &= (1 - \delta) k_t + sy_t \\ &= (1 - \delta) k_t + sAk_t^{\theta+\eta} H_t^\eta \end{aligned}$$

- to get

$$\gamma_t = \frac{k_{t+1}}{k_t} - 1 = sAk_t^{\theta+\eta-1} H_t^\eta - \delta$$

- In the AK case where $\theta + \eta = 1$, we get

$$\gamma_t = sAH_t^\eta - \delta$$

- H_t is the number of workers in the country, so bigger countries (China, India, USA, Mexico, Brazil) should grow faster than smaller ones (Hong Kong, Singapore, Luxemburg). This is a model with a *scale effect*.

Optimal taxes where government expenditures enter the production function

- Suppose government expenditures, G_t , enter the production function as

$$Y_t = AK_t^\theta G_t^{1-\theta}$$

- Disposable income (which is what matters to households) is

$$Y_t^d = (1 - \tau) Y_t = (1 - \tau) AK_t^\theta G_t^{1-\theta}$$

where τ is the tax rate on output.

- Assume that the government's budget is balanced: taxes equal expenditures, so

$$\tau Y_t = G_t$$

Optimal taxes

- Writing everything in per worker terms (dividing by H_t) and defining $g_t = G_t/H_t$, one gets

$$y_t^d = (1 - \tau) Ak_t^\theta g_t^{1-\theta}$$

- The law of motion of capital is

$$\begin{aligned} (1 + n) k_{t+1} &= (1 - \delta) k_t + s y_t^d \\ &= (1 - \delta) k_t + s (1 - \tau) Ak_t^\theta g_t^{1-\theta} \end{aligned}$$

- The growth rate of capital is

$$(1 + n) \gamma_t = (1 + n) \left[\frac{k_{t+1}}{k_t} - 1 \right] = s (1 - \tau) Ak_t^{\theta-1} g_t^{1-\theta} - (n + \delta)$$

Optimal taxes

- Using the government's budget constraint, one gets

$$g_t = \tau y_t = \tau Ak_t^\theta g_t^{1-\theta}$$

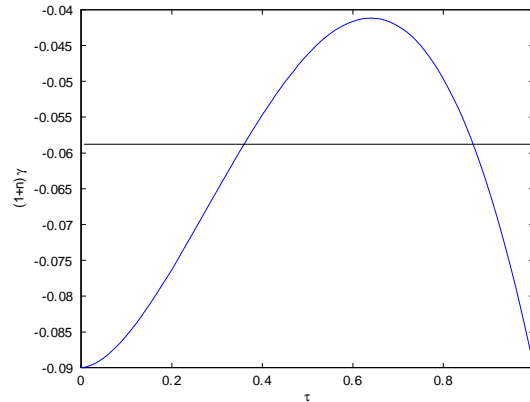
which can be solved to get

$$g_t = \tau^{\frac{1}{\theta}} A^{\frac{1}{\theta}} k_t$$

- Putting this into the growth rate equation gives

$$\begin{aligned} (1 + n) \gamma_t &= s (1 - \tau) Ak_t^{\theta-1} \left(\tau^{\frac{1}{\theta}} A^{\frac{1}{\theta}} k_t \right)^{1-\theta} - (n + \delta) \\ &= s (1 - \tau) A A^{\frac{1-\theta}{\theta}} \tau^{\frac{1-\theta}{\theta}} - (n + \delta) \\ &= s (1 - \tau) A^{\frac{1}{\theta}} \tau^{\frac{1-\theta}{\theta}} - (n + \delta) \end{aligned}$$

Optimal taxes



- The optimal tax rate can be found from $\frac{d(1+n)\gamma_t}{d\tau} = 0$, or

$$0 = -sA^{\frac{1}{\theta}}\tau^{\frac{1-\theta}{\theta}} + \frac{1-\theta}{\theta}s(1-\tau)A^{\frac{1}{\theta}}\tau^{\frac{1-\theta}{\theta}-1}$$

- or

$$\frac{\theta}{1-\theta} = \frac{(1-\tau)}{\tau}$$

- or

$$\tau^* = 1 - \theta$$

Optimal taxes

- For an economy with $\theta = .36$, $A = 1$, $s = .3$, $\delta = .08$, $n = .01$, the growth rate as a function of the tax is
- max is at $\tau = .64$

Homework

- Consider a model of learning by doing where the output of period t is a function of the output of period $t - 1$ of the form

$$y_t = Ak_t^\theta y_{t-1}^\eta$$

- Describe the economy when $\theta + \eta < 1$
 - find the stationary states
- Describe the economy when $\theta + \eta = 1$
 - find the constant growth rate for an example economy (you might need to use Matlab)