Crecimiento Economico
Modelos de Generaciones sobrepuestos
Bonos y produccion

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Bonos del gobierno

- bonos reales: pagan $B(t)$ unidades de bienes en periodo t+1
- Son bonos de 1 periodo
- El gobierno puede pagar estos bonos con
  - impuestos sobre los jovenes de periodo $t+1$
  - impuestos sobre los viejos de period $t+1$
  - emitir nuevo bonos en periodo $t+1$ ($B(t+1)$)
  - una combinación de estos tres
- Restricción de presupuesto en periodo t de gobierno con bonos

$$0=\sum_{h=1}^{N(t)} t_t^h\left(t\right)+\sum_{h=1}^{N(t-1)} t_{t-1}^h\left(t\right)-B(t-1)+p(t)B(t)$$

donde $p(t)$ es el precio de nuevo bonos en periodo t

Restricciones de presupuesto de los individuos con bonos

- Restricciones de los jovenes

$$c_t^h\left(t\right)=w_t^h\left(t\right)-t_t^h\left(t\right)-l^h\left(t\right)-p(t)b^h(t)$$

cuando joven, y

$$c_t^h\left(t+1\right)=w_t^h\left(t+1\right)-t_t^h\left(t+1\right)+r(t)l^h\left(t\right)+b^h(t)$$

cuando viejos

- Restricción de presupuesto de la vida

$$c_t^h(t) + \frac{c_t^h(t+1)}{r(t)} = w_t^h(t) + \frac{w_t^h(t+1)}{r(t)} - b^h(t)\left(p(t) - \frac{1}{r(t)}\right)$$

- la misma restriccion de antes pero con

$$-b^h(t)\left(p(t) - \frac{1}{r(t)}\right)$$

.Condición de no arbitraje

- Que valor debe tener

$$p(t) - \frac{1}{r(t)}$$

- Tres posibilidades

$$\begin{aligned} p(t) - \frac{1}{r(t)} &< 0 \\ p(t) - \frac{1}{r(t)} &= 0 \\ p(t) - \frac{1}{r(t)} &> 0 \end{aligned}$$

.Condición de no arbitraje

- condición de no arbitraje es

$$p(t) = \frac{1}{r(t)}$$

y retorno en bonos es igual a retorno en prestamos privados

.Condiciòn de equilibrio en mercado de bonos del gobierno

- Suma restricciones de presupuesto de los jovenes de periodo t

$$\sum_{h=1}^{N(t)} c_t^h(t) = \sum_{h=1}^{N(t)} w_t^h(t) - \sum_{h=1}^{N(t)} t_t^h(t) - \sum_{h=1}^{N(t)} l^h(t) - p(t)\sum_{h=1}^{N(t)} b^h(t)$$

- Nota: en equilibrio

$$\sum_{h=1}^{N(t)} l^h(t) = 0$$

$$\sum_{h=1}^{N(t)} b^h(t) = B(t)$$

y

$$\sum_{h=1}^{N(t)} w_t^h(t) - \sum_{h=1}^{N(t)} t_t^h(t) - \sum_{h=1}^{N(t)} c_t^h(t) = S(r(t))$$

- Entonces
$$S(r(t)) = p(t)B(t)$$

Equilibrio en mercado de bonos del gobierno

- Condición de "market clearing":
$$S(r(t)) = p(t)B(t)$$
y condición de no arbitraje
$$p(t) = \frac{1}{r(t)}$$

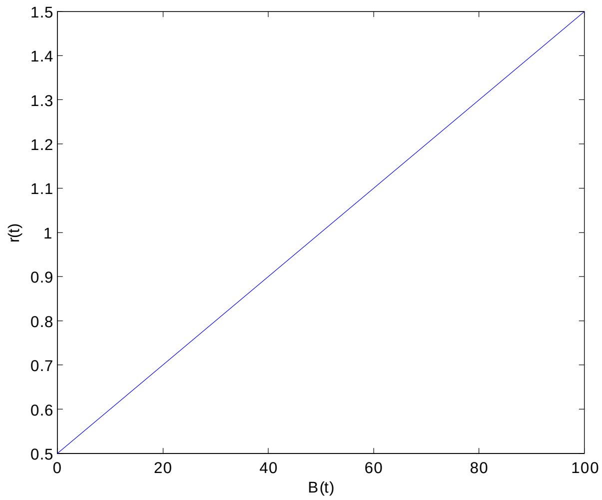
- Busca soluciòn (un $r(t)$) donde
$$S(r(t)) = \frac{1}{r(t)}B(t)$$

Ejemplo:

- Gobierno emite 10 bonos en periodo 1 y usa los ingresos para trasferencias a miembros de generación 0 (los viejos). Cobra impuesto de los jovenes de generación 2 para pagar los bonos. ¿Cómo es el equilibrio por los miembros de generación 1? $N(1) = 100$, $w_1^h = [2, 1]$, y $\beta = 1$.
- Solución: función de ahorros agregados es
$$S(r(t)) = 100 - \frac{50}{r(t)}$$
- Condición de equilibrio implica
$$100 - \frac{50}{r(t)} = \frac{1}{r(t)}10$$
- Resuelve y
$$r(t) = 0,6$$

Tasa de interés como función de numero de bonos
Caracteristicas especiales del ejemplo

1. Ingreso de los bonos va a los viejos de generación 0
2. Los joven de generación 2 paga impuestos para cubrir los bonos
3. Miembros de generación 1 compran y verden los bonos pero no pagan ni impuestos y reciben trasferencias. Mercado de bonos funciona solo para miembros de generación 1.



."Rolling over" bonos

- En periodo 2, el gobierno podría vender nuevos bonos (y no cobrar impuestos)
- Debe vender sufficiente bonos para tener $B(1)$ para pagar bonos de periodo 1.
- Ingresos de venta de bonos es $p(2)B(2)$
- El gobierno quiere vender $B(2)$ bonos donde

$$p(2)B(2) = B(1)$$

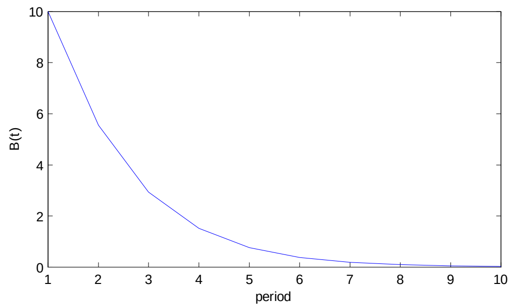
- Si quiere "rolling over", en periodo 3 debe vender $B(3)$ para que

$$p(3)B(3) = B(2)$$

- Nota: en esto ejemplo solo los viejos de generación 0 recibe una trasferencia

.Un ejemplo de "rolling over"

- Gobierno emite 10 bonos en periodo 1 y usa los ingresos para trasferencias a miembros de generación 0 (los viejos). Cobra impuesto de los jovenes de generación 2 para pagar los bonos. ¿Cómo es el equilibrio? $N(t) = 100$, $w_t^h = [2, 1]$, y $\beta = 1$.
- Solución



– Periodo 1 es como antes:

$$S(r(1)) = p(1)B(1)$$

or

$$100 - \frac{50}{r(1)} = \frac{1}{r(1)}10$$

.

- En periodo 2,

$$p(2)B(2) = B(1) = 10$$

o

$$S(r(2)) = 10$$

y

$$100 - \frac{50}{r(2)} = 10$$

– Entonces

$$r(2) = \frac{50}{90} = \frac{5}{9}$$

y

$$B(2) = \frac{10}{p(2)} = \frac{50}{9}$$

.

- En periodo 3,

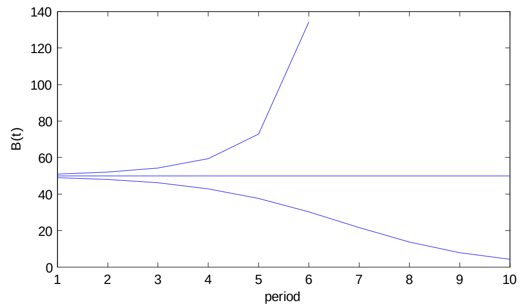
$$p(3)B(3) = B(2) = \frac{50}{9}$$

- Resuelve y

$$B(3) = \frac{50}{17}$$

- Sigiendo

.Camino de "rolling over" como función de X



- Si $X > 50$, no hay equilibrio con "rolling over"

¿Que es especial sobre 50 (en esto ejemplo)?

- $B(1) = 50$
- Solución
  - Periodo 1 es

$$S(r(1)) = p(1)B(1)$$

o

$$100 - \frac{50}{r(1)} = \frac{1}{r(1)}50$$

- $$r(1) = 1$$

y

$$p(1) = \frac{1}{r(1)} = 1$$

- Periodo 2: gobierno debe emitir bonos para tener ingreso = 50
- emitiendo 50 bonos en periodo 2 y

$$r(2) = 1 \text{ y } p(2) = \frac{1}{r(2)} = 1$$

¿Que es especial sobre 50 (en esto ejemplo)?

- Con $B(1) = 50$ (en esta economia)
  - $B(t) = 50, \forall t$
  - $\left[c_t^h(t), c_t^h(t+1)\right] = \left[\frac{3}{2}, \frac{3}{2}\right], \forall t > 1$
  - $c_0^h(1) = \frac{3}{2}$
  - Es optima de Pareto

Model with production

- We use the same basic model but change endownments from goods endownments to labor endownments
- Both the young and old supply labor
- The young buy (save) capital into the next period
- They earn the rent on that capital
- If the young do not save, there is no production in the next period

The Environment

- Time is discrete: $t = 0, 1, 2, 3, 4, ...$
- Individuals live 2 periods
  - Generation t is born in period t
  - Lives in periods t and $t + 1$
  - They are dead in periods $t + 2$ and onwards
  - There are $N(t)$ members of generation t
- Individuals have preferences: person h of generation t

$$u_t^h(c_t^h(t), c_t^h(t+1))$$

- Individuals have a **labor** endownment:
- no labor supply decision, they supply all

$$h_t^h = \left[h_t^h(t), h_t^h(t+1)\right]$$

The Environment

- Production technology

$$Y(t) = F(K(t), H(t))$$

- Total labor supplied in period t

$$H(t) = \sum_{h=1}^{N(t)} h_t^h(t) + \sum_{h=1}^{N(t-1)} h_{t-1}^h(t)$$

Useful to defind labor supplied by each generation in period t
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5cm

$$H_t(t) = \sum_{h=1}^{N(t)} h_t^h(t)$$

5cm

$$H_{t-1}(t) = \sum_{h=1}^{N(t-1)} h_{t-1}^h(t)$$

- Feasiblity constraint for period t

$$Y(t) = F(K(t), H(t)) \geq \sum_{h=1}^{N(t)} c_t^h(t) + \sum_{h=1}^{N(t-1)} c_{t-1}^h(t) + K(t+1)$$

The economy (a market economy)

- Budget constraint of generation t when young

$$w_t h_t^h(t) = c_t^h(t) + l^h(t) + k^h(t+1)$$

  - w_t are the wages paid at time t
  - $l^h(t)$ is lending or borrowing (if negative) of young in period t
  - Can only lend to own generation (why?)
  - so

$$0 = \sum_{h=1}^{N(t)} l_h(t)$$

- Budget constaint of generation t when old

$$c_t^h(t+1) = w_{t+1} h_t^h(t+1) + r_t l^h(t) + rental_{t+1} k^h(t+1)$$

  - r_t is the interest rate on private loans
  - $rental_{t+1}$ is the rental rate on capital when it is being used
  - here we assume that capital depreciates 100% in the period of use

Factor market conditions

- Perfectly compeditive factor markets
- So

$$w_t = F_H(K(t), H(t))$$

  and

$$rental_t = F_K(K(t), H(t))$$

- Factor rentals equal their marginal products

Lifetime budget constraint

- The lifetime budget constraint is

$$\begin{aligned} c_t^h(t) + \frac{c_t^h(t+1)}{r_t} &= w_t h_t^h(t) + \frac{w_{t+1} h_t^h(t+1)}{r_t} \\ &\quad - k^h(t+1)\left[1 - \frac{rental_{t+1}}{r_t}\right] \end{aligned}$$

No Arbitrage conditions

- Arbitrage condition (actually NO arbitrage condition)

$$rental_{t+1} = r_t$$

  - What if not true: two options $rental_{t+1} < r_t$ or $rental_{t+1} > r_t$
  - If $rental_{t+1} < r_t$
    * Everyone wants to lend and no one holds capital
    * But marginal product of capital $\longrightarrow \infty$ when $K(t) \longrightarrow 0$
    * Not an equilibrium
  - If $rental_{t+1} > r_t$
    * Everyone wants to borrow an infinite amount to buy capital
    * This can't be an equilibrium
  - Only $rental_{t+1} = r_t$ remains as possibility

Lifetime budget constraint II

- The lifetime budget constraint simplifies to

$$c_t^h(t) + \frac{c_t^h(t+1)}{r_t} = w_t h_t^h(t) + \frac{w_{t+1} h_t^h(t+1)}{r_t}$$

- The present value of lifetime consumption equals the present value of lifetime wage income

Definition of an equilibrium

A *competitive equilibrium* is a sequence of prices, $\{w_t, rental_t, r_t\}_{t=0}^{\infty}$, and quantities, $\left\{\left\{c_t^h(t)\right\}_{h=1}^{N(t)}, \left\{c_{t-1}^h(t)\right\}_{h=1}^{N(t-1)}, K(t+1)\right\}_{t=0}^{\infty}$, such that each member h of each generation $t > 0$ maximizes the utility function,

$$u_t^h(c_t^h(t), c_t^h(t+1))$$

subject to the lifetime budget constraint,

$$c_t^h(t) + \frac{c_t^h(t+1)}{r_t} = w_t h_t^h(t) + \frac{w_{t+1} h_t^h(t+1)}{r_t}$$

and the equilibrium conditions,
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$$\begin{aligned} r_t &= rental_{t+1} \\ w_t &= F_H(K(t), H(t))) \\ rental_t &= F_K(K(t), H(t)) \end{aligned}$$

5cm

$$H(t) = \sum_{h=1}^{N(t)} h_t^h(t) + \sum_{h=1}^{N(t-1)} h_{t-1}^h(t)$$

hold in every period.
How to solve: Individual problem

- Substitute the budget constraints into the utility function to get

$$\max_{c_t^h(t)} u(c_t^h(t), r_t w_t h_t^h(t) + w_{t+1} h_t^h(t+1) - r_t c_t^h(t))$$

- First order condtions are

$$\begin{aligned} & u_1(c_t^h(t), r_t w_t h_t^h(t) + w_{t+1} h_t^h(t+1) - r_t c_t^h(t)) \\ = \quad & r_t u_2(c_t^h(t), r_t w_t h_t^h(t) + w_{t+1} h_t^h(t+1) - r_t c_t^h(t)), \end{aligned}$$

- One can solve for a savings functions as

$$s_t^h(w_t, w_{t+1}, r_t) = l^h(t) + k^h(t+1).$$

How to solve: Aggregating savings functions

- Sum savings across members of generation t:

$$S_t(\cdot) = \sum_{h=1}^{N(t)} s_t^h(\cdot) = \sum_{h=1}^{N(t)} l^h(t) + \sum_{h=1}^{N(t)} k^h(t+1).$$

- In equilibrium, total borrowing and lending among generation t is

$$\sum_{h=1}^{N(t)} l^h(t) = 0,$$

- Definition of aggregate capital is

$$K(t+1) = \sum_{h=1}^{N(t)} k^h(t+1),$$

- So the aggregate savings equation is

$$S_t(w_t, w_{t+1}, r_t) = K(t+1).$$

How to solve: Getting a first order difference equation

- From the factor markets we have wages and rentals in terms of capital and labor,

$$\begin{aligned} w_t &= F_H(K(t), H(t))) \\ rental_t &= F_K(K(t), H(t)) \end{aligned}$$

- Substitute these into the aggregate saving equation to get (this ugly equation)

$$S_t(F_H(K(t), H(t)), F_H(K(t+1), H(t+1)), F_K(K(t+1), H(t+1)))$$
$$= K(t+1)$$

- This can be simplified (since $H(t)$ and $H(t+1)$ are constants) to get a function of the form

$$K(t+1) = G(K(t))$$

An example economy

- Let the utility function be

$$u_t^h = u(c_t^h(t), c_t^h(t+1)) = c_t^h(t) c_t^h(t+1)^\beta$$

and the production function be

$$Y_t = F(K(t), H(t)) = K(t)^\theta H(t)^{1-\theta}$$

.

- The first order condtions are

$$\begin{aligned} & u_1(c_t^h(t), r_t w_t h_t^h(t) + w_{t+1} h_t^h(t+1) - r_t c_t^h(t)) \\ = \quad & r_t u_2(c_t^h(t), r_t w_t h_t^h(t) + w_{t+1} h_t^h(t+1) - r_t c_t^h(t)), \end{aligned}$$

- For the utility function we are using

$$u_t^h = c_t^h(t) c_t^h(t+1)^{\beta},$$

this gives

$$c_t^h(t+1)^{\beta} = \beta r_t c_t^h(t) c_t^h(t+1)^{\beta-1}$$

or

$$c_t^h(t+1) = \beta r_t c_t^h(t)$$

or

$$r_t w_t h_t^h(t) + w_{t+1} h_t^h(t+1) - r_t c_t^h(t) = \beta r_t c_t^h(t)$$

which gives the individual consumption function

$$c_t^h(t) = \frac{r_t w_t h_t^h(t)}{(1+\beta)\, r_t} + \frac{w_{t+1} h_t^h(t+1)}{(1+\beta)\, r_t}$$

.

- The individual savings function is

$$\begin{aligned} s_t^h(w_t, w_{t+1}, r_t) &= w_t h_t^h(t) - \frac{r_t w_t h_t^h(t)}{(1+\beta)\, r_t} - \frac{w_{t+1} h_t^h(t+1)}{(1+\beta)\, r_t} \\ &= l^h(t) + k^h(t+1), \end{aligned}$$

or

$$s_t^h(w_t, w_{t+1}, r_t) = \frac{\beta w_t h_t^h(t)}{(1+\beta)} - \frac{w_{t+1} h_t^h(t+1)}{(1+\beta)\, r_t} = l^h(t) + k^h(t+1).$$

- Summing over the members of generation h, we get

$$S_t(w_t, w_{t+1}, r_t) = \frac{\beta w_t H_t(t)}{(1+\beta)} - \frac{w_{t+1} H_t(t+1)}{(1+\beta)\, r_t} = K(t+1).$$

.

- From the conditions of the factor markets and the no arbitrage condition, we know that

$$\begin{aligned} r_t &= rental_{t+1} \\ w_t &= (1-\theta)\, K(t)^{\theta} H(t)^{-\theta} \\ rental_t &= \theta K(t)^{\theta-1} H(t)^{1-\theta} \end{aligned}$$

- The aggregate savings function can be written as

$$\begin{aligned} & \frac{\beta w_t H_t(t)}{(1+\beta)} + \frac{w_{t+1} H_t(t+1)}{(1+\beta)\, r_t} \\ = \quad & \frac{\beta (1-\theta)\, K(t)^{\theta} H(t)^{-\theta} H_t(t)}{(1+\beta)} \\ & - \frac{(1-\theta)\, K(t+1)^{\theta} H(t+1)^{-\theta} H_t(t+1)}{(1+\beta)\, \theta K(t+1)^{\theta-1} H(t+1)^{1-\theta}} \end{aligned}$$

.

- In equilibrium

$$\begin{aligned} & \frac{\beta(1-\theta) K(t)^\theta H(t)^{-\theta} H_t(t)}{(1+\beta)} \\ & -\frac{(1-\theta) K(t+1)^\theta H(t+1)^{-\theta} H_t(t+1)}{(1+\beta) \theta K(t+1)^{\theta-1} H(t+1)^{1-\theta}} \\ = & K(t+1) \end{aligned}$$

- or

$$\begin{aligned} & \frac{\beta(1-\theta) K(t)^\theta H_t(t)}{(1+\beta) H(t)^\theta} \\ & -\frac{(1-\theta) K(t+1) H_t(t+1)}{(1+\beta) \theta H(t+1)} \\ = & K(t+1) \end{aligned}$$

.

- After a bit of algebra, the above expression becomes

$$K(t+1)=\frac{\theta \beta \frac{H_t(t)}{H(t)^\theta}}{\left[\frac{H_t(t+1)}{H(t+1)}\right]+\frac{\theta(1+\beta)}{(1-\theta)}} K(t)^\theta=\kappa K(t)^\theta$$

where κ is a constant equal to

$$\kappa=\frac{\theta \beta \frac{H_t(t)}{H(t)^\theta}}{\left[\frac{H_t(t+1)}{H(t+1)}\right]+\frac{\theta(1+\beta)}{(1-\theta)}}$$

.The $K(t+1)=G(K(t))$ function is
.Model with growth in technology

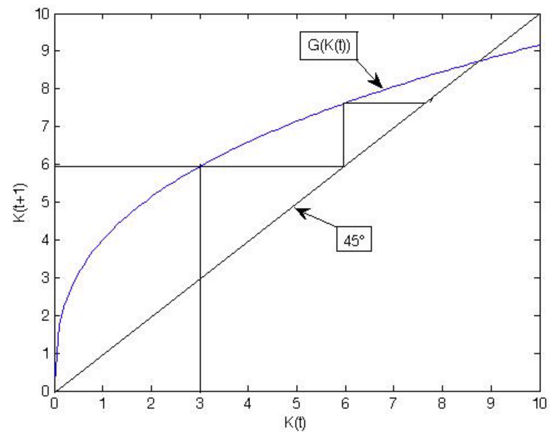
- What happens if $A_t=(1+a)^t A_0$ in the production function

$$\begin{aligned} Y_t & =A_t K_t^\theta H_t^{1-\theta} \\ & =(1+a)^t A_0 K_t^\theta H_t^{1-\theta} . \end{aligned}$$

- The equilibrium with respect to the aggregate saving function is still

$$S_t(w_t, w_{t+1}, r_t)=\frac{\beta w_t H_t(t)}{(1+\beta)}-\frac{w_{t+1} H_t(t+1)}{(1+\beta) r_t}=K(t+1).$$

.



- The no arbitrage condition is still

$$r_t = rental_{t+1},$$

but now the factor market conditions are

$$w_t = (1-\theta)\,(1+a)^t A_0 K(t)^\theta H(t)^{-\theta}$$

and

$$rental_t = \theta(1+a)^t A_0 K(t)^{\theta-1} H(t)^{1-\theta}.$$

.

- Substituting these into the equilibrium condition gives

$$\begin{aligned} K(t+1) &= \frac{\beta\,(1-\theta)\,(1+a)^t A_0 K(t)^\theta H(t)^{-\theta} H_t(t)}{(1+\beta)} \\ &\quad - \frac{(1-\theta)\,(1+a)^{t+1} A_0 K(t+1)^\theta H(t+1)^{-\theta} H_t(t+1)}{(1+\beta)\,\theta(1+a)^{t+1} A_0 K(t+1)^{\theta-1} H(t+1)^{1-\theta}} \end{aligned}$$

- which can be simplified to

$$\begin{aligned} K(t+1) &= (1+a)^t \frac{\beta\,(1-\theta)\,A_0 H_t(t)}{(1+\beta)\,H(t)^\theta} K(t)^\theta \\ &\quad - \frac{(1-\theta)\,H_t(t+1)}{(1+\beta)\,\theta H(t+1)} K(t+1) \end{aligned}$$

.

- and to

$$K(t+1) = (1+a)^t A_0 \frac{\beta\theta \frac{H_t(t)}{H(t)^\theta}}{\left[\frac{(1+\beta)\theta}{(1-\theta)} + \frac{H_t(t+1)}{H(t+1)}\right]} K(t)^\theta.$$

- Define the constant κ as before,

$$\kappa = \frac{\beta\theta\frac{H_t(t)}{H(t)^\theta}}{\left[\frac{(1+\beta)\theta}{(1-\theta)} + \frac{H_t(t+1)}{H(t+1)}\right]},$$

- and the model can be written as

$$K(t+1) = (1+a)^t A_0 \kappa K(t)^\theta.$$

Balanced growth path

- For a balanced growth path (where the growth rate of capital is constant), we look for

$$\gamma = \frac{K(t+1)}{K(t)} = (1+a)^t A_0 \kappa K(t)^{\theta-1},$$

- or

$$K(t) = \left[\frac{A_0\kappa(1+a)^t}{\gamma}\right]^{\frac{1}{1-\theta}},$$

- or

$$K(t) = \left[\frac{A_0\kappa}{\gamma}\right]^{\frac{1}{1-\theta}} (1+a)^{\frac{t}{1-\theta}},$$

.

- $K(0)$ must be equal to

$$K(0) = \left[\frac{A_0\kappa}{\gamma}\right]^{\frac{1}{1-\theta}},$$

- and

$$\gamma = \frac{K(t+1)}{K(t)} = \frac{\left[\frac{A_0\kappa}{\gamma}\right]^{\frac{1}{1-\theta}} (1+a)^{\frac{t+1}{1-\theta}}}{\left[\frac{A_0\kappa}{\gamma}\right]^{\frac{1}{1-\theta}} (1+a)^{\frac{t}{1-\theta}}} = (1+a)^{\frac{1}{1-\theta}}$$

.

Graph of balanced growth path in basic Solow model

What happens if depreciation is not 100%?

- The old sell their remaining capital to the young
- The young save based on the return on capital (which includes selling it)
- Budget constraint of young is the same as before

$$w_t h_t^h(t) = c_t^h(t) + l^h(t) + q_t k^h(t+1)$$

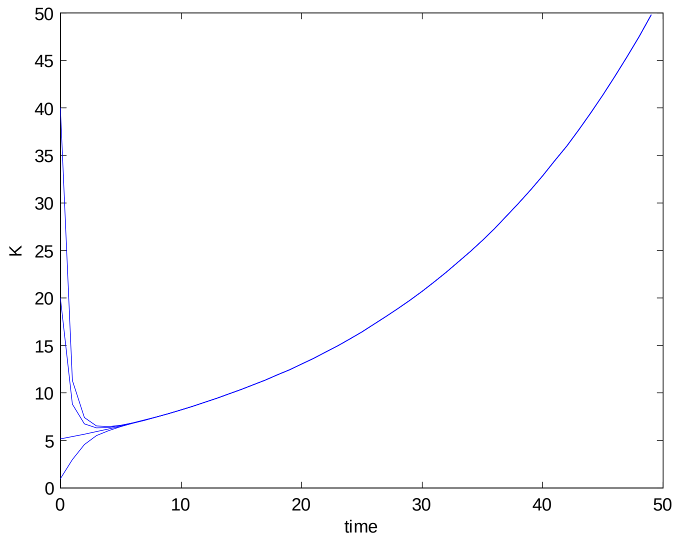
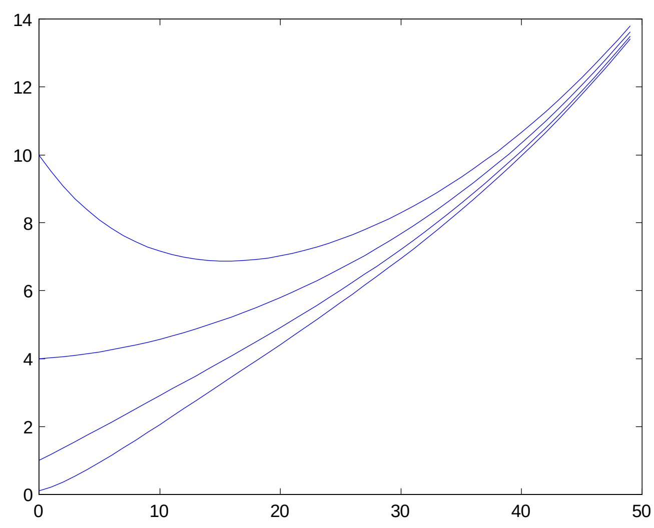


Figure 1: Growth path with technological change



- Budget constaint of the old is

$$\begin{aligned} c_t^h(t+1) &= w_{t+1}h_t^h(t+1) + r_t l^h(t) + rental_{t+1}k^h(t+1) \\ &\quad + (1-\delta)\, q_{t+1}k^h(t+1) \end{aligned}$$

- where δ is the depreciation rate and q_t is the price of capital goods in period t
- $q_t = 1$ if $k^h(t+1) \geq (1-\delta)\, k^h(t)$ because consumption good gets converted to capital goods at a 1 to 1 rate
  - since old capital is identical to new, it needs to have a price of 1
- rest of the model follows