Crecimiento economico
Class 13
Multiple equilibrium (sunspots)

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.Sunspots

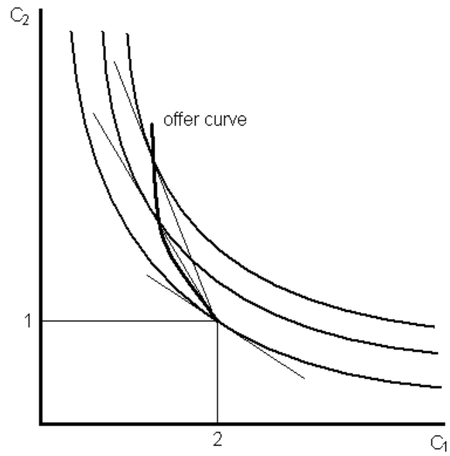
- Not exactly growth, but important in dynamic models
  - can imply multiple growth paths possible
- Issue of multiple nonstationary equilibria
- Models of Caotic behavior
- That Caotic behavior comes from a low dimension predictable model
- The famous "butterfly effect"
- Note: expectations will be perfect foresight (limit of rational expectations)

.What we do

- Begin with a standard OLG model
- Change the utility function
- This changes behavior
- results in an economy with multiple equilibria
- The model we use is OLG with land

.OLG with land

- The economy has $N(t) = 100$ people born per period
- These people live two periods: period t and $t + 1$
- The are born with a goods endowment: $w_t^h = [2, 1]$



- There are $\mathbf{A} = 100$ units of land in the economy
- Each unit of land produces $d(t)$ units of crop in period t
  - $d(t)$ is constant through time
  - we begin with a model where $d(t) = 0$
- all members of generation t have the same utility function u_t
  - in the previous OLG models, $u_t^h = c_t^h(t) \cdot c_t^h(t+1)^{\beta}$
  - a standard Cobb-Douglas type utility function
  - indifference curves are rectangular hyperbola

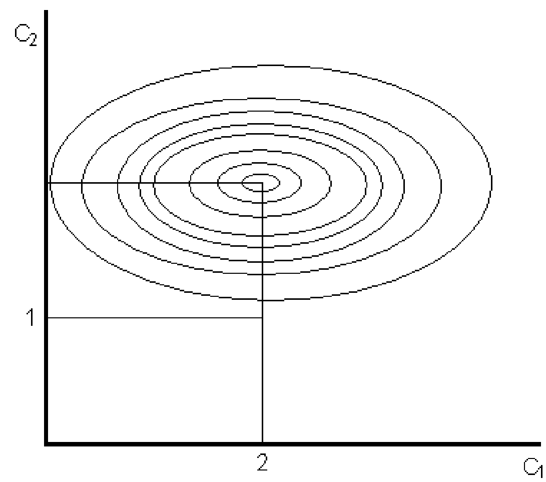
.Cobb-Douglas utility functions and offer curve
.A different utility function

- Here we use the utility function

$$u_t^h = -\left(c_t^h(t) - b^y\right)^2 - \beta\left(c_t^h(t+1) - b^o\right)^2$$

- The version we will use has $\beta = 4$, and $[b^y, b^o] = [2, 2]$
- what is special about this utility function

.Indifference curves of new utility function
.Offer curve of new utility function
.Offer curve and elasticity of demand for consumption in period 2
.Characteristic of all models with sunspots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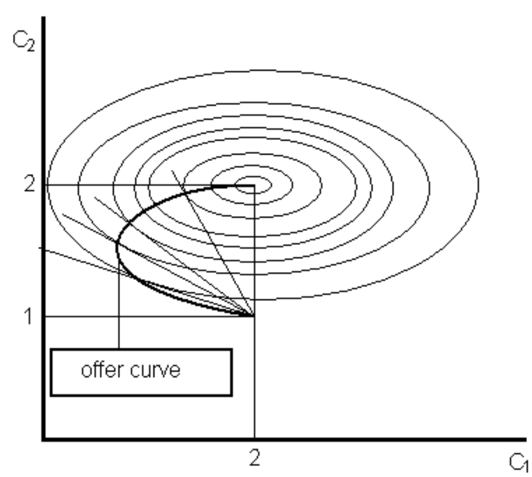
C2
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offer curve
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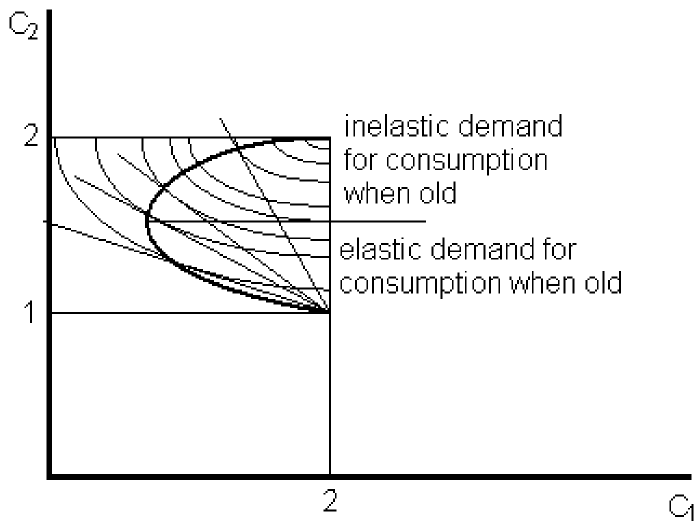


- The offer curve has a inelastic section
- It need not be as severe as in this model
- The backwards bending part is what generates multiple equilibria

Back to the model

- Budget constraint when young

$$c_t^h(t) = w_t^h(t) - l^h(t) - p(t)a^h(t)$$

- $a^h(t)$ is the amount of land that agent h buys
- $p(t)$ is the time t price of a unit of land
- Budget constraint when old

$$c_t^h(t+1) = w_t^h(t+1) + r(t)l^h(t) - [p(t+1) + d(t+1)]\,a^h(t)$$

- $d(t+1)$ is the crop that a unit of land will produce in period $t+1$

Lifetime budget constraint

- Combine the two budget constraints by solving for $l^h(t)$

- Get

$$\begin{aligned} c_t^h(t) + \frac{c_t^h(t+1)}{r(t)} &= w_t^h(t) + \frac{w_t^h(t+1)}{r(t)} \\ &\quad - a^h(t)\left[p(t) - \frac{p(t+1)+d(t+1)}{r(t)}\right] \end{aligned}$$

- No arbitrage condition gives

$$p(t) = \frac{p(t+1)+d(t+1)}{r(t)}$$

- Current price of the land is the present value of the future price plus crop
- Why is this an arbitrage condition?

Lifetime budget constraint

- The no-arbitrage condition says that the part in brackets equals zero, so the lifetime budget constraint is

$$c_t^h(t) + \frac{c_t^h(t+1)}{r(t)} = w_t^h(t) + \frac{w_t^h(t+1)}{r(t)}$$

- Want to maximize

$$u_t^h = -\left(c_t^h(t) - b^y\right)^2 - \beta\left(c_t^h(t+1) - b^o\right)^2$$

subject to that budget constraint

- get the Lagrangian

$$\begin{aligned} \mathcal{L} &= -\left(c_t^h(t) - b^y\right)^2 - \beta\left(c_t^h(t+1) - b^o\right)^2 \\ &\quad + \lambda\left[c_t^h(t) + \frac{c_t^h(t+1)}{r(t)} - w_t^h(t) - \frac{w_t^h(t+1)}{r(t)}\right] \end{aligned}$$

- Maximize with respect to $c_t^h(t)$ and $c_t^h(t+1)$

Conditions for an equilibrium

- The first order conditions from the Lagrangian are

$$2\left(c_t^h(t) - b^y\right) = \lambda$$

and

$$2\beta\left(c_t^h(t+1) - b^o\right) = \lambda\frac{1}{r(t)}$$

- The budget constraint is

$$c_t^h(t) + \frac{c_t^h(t+1)}{r(t)} = w_t^h(t) + \frac{w_t^h(t+1)}{r(t)}$$

- The no-arbitrage condition is

$$p(t) = \frac{p(t+1) + d(t+1)}{r(t)}$$

.The consumption function

- The two first order conditions can be simplified to

$$c_t^h(t+1) = \frac{\left(c_t^h(t) - b^y\right)}{r(t)\beta} + b^o$$

- Putting that into a consumption function of

$$c_t^h(t) = w_t^h(t) + \frac{w_t^h(t+1)}{r(t)} - \frac{c_t^h(t+1)}{r(t)}$$

to get

$$c_t^h(t) = w_t^h(t) + \frac{w_t^h(t+1) - b^o}{r(t)} - \frac{c_t^h(t) - b^y}{r(t)^2\beta}$$

or

$$c_t^h(t)\left(1 + \frac{1}{r(t)^2\beta}\right) = w_t^h(t) + \frac{w_t^h(t+1) - b^o}{r(t)} + \frac{b^y}{r(t)^2\beta}$$

.The consumption function

- or

$$c_t^h(t) = \frac{r(t)^2\beta w_t^h(t)}{r(t)^2\beta + 1} + \frac{\left(w_t^h(t+1) - b^o\right)r(t)\beta}{r(t)^2\beta + 1} + \frac{b^y}{r(t)^2\beta + 1}$$

.The savings function

- Savings is equal to

$$s^h(t) = w_t^h(t) - c_t^h(t)$$

- after substituting in the consumption function

$$s^h(t) = \frac{w_t^h(t) - b^y - \beta r(t)\left(w_t^h(t+1) - b^o\right)}{1 + \beta r(t)^2}$$

- Aggregate savings comes from summing the savings of the young to get

$$\begin{aligned} S_t\left(r(t)\right) &= \sum_{h=1}^{N(t)} \left(w_t^h\left(t\right) - c_t^h\left(t\right)\right) = \sum_{h=1}^{N(t)} \left(l^h\left(t\right) + p(t)a^h\left(t\right)\right) \\ &= \sum_{h=1}^{N(t)} p(t)a^h\left(t\right) = p(t)\sum_{h=1}^{N(t)} a^h\left(t\right) = p(t)\mathbf{A} \end{aligned}$$

- So in equilibrium

$$S_t\left(r(t)\right) = p(t)\mathbf{A}$$

.The example economy

- All members of a generation are alike, so

$$S_t\left(r(t)\right) = N(t)s^h\left(t\right) = N(t)\frac{w_t^h\left(t\right) - b^y - \beta r(t)\left(w_t^h\left(t+1\right) - b^o\right)}{1 + \beta r(t)^2}$$

- For the specific economy of the model, with $\beta = 4$, $N(t) = 100$, $[b^y, b^o] = [2, 2]$ and $w_t^h = [2, 1]$, the aggregate savings function is

$$\begin{aligned} S_t\left(r(t)\right) &= 100\frac{2 - 2 - \beta r(t)\left(1 - 2\right)}{1 + \beta r(t)^2} \\ &= 100\frac{\beta r(t)}{1 + \beta r(t)^2} \end{aligned}$$

.The equilibrium for the example economy

- Putting the savings function and the fact that $\mathbf{A} = 100$ into the equilibrium condition, one gets

$$100\frac{\beta r(t)}{1 + \beta r(t)^2} = 100p(t)$$

- which simplifies to

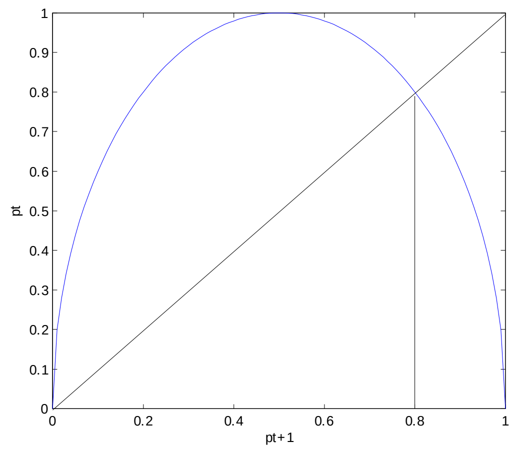
$$\frac{\beta r(t)}{1 + \beta r(t)^2} = p(t)$$

- Use the no-arbitrage condition as

$$r(t) = \frac{p(t+1) + d(t+1)}{p(t)}$$

- to get

$$\frac{\beta\left[\frac{p(t+1)+d(t+1)}{p(t)}\right]}{1 + \beta\left[\frac{p(t+1)+d(t+1)}{p(t)}\right]^2} = p(t)$$



.The equilibrium for the example economy

- After a bit of algebra, one gets

$$p(t) = \left[4p\,(t+1) - 4p\,(t+1)^2\right]^{\frac{1}{2}}$$

- which is the classic equation from chaos theory
- This gives the time t price of land as a function of the expected (and with perfect foresight, the realized) price in period $t+1$

.Graph of price function

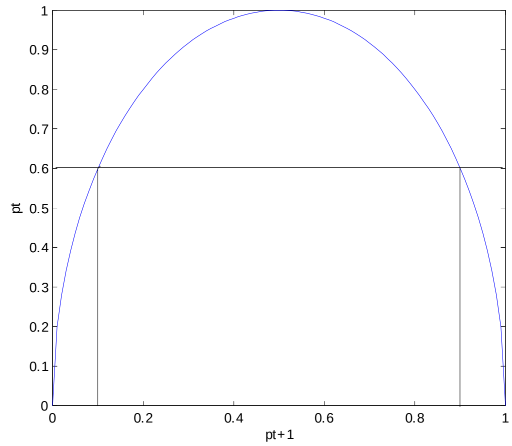
- The stationary states are at $p(t) = p(t+1) = 0$ and $p(t) = p(t+1) = .8$

.Other equilibria

- Are there other equilibria?
- An equilibrium is a path of $p(t)$ that never violates any of the conditions of the model
- That means that the prices $p(t{+}1)$, $p(t{+}2)$,... that follow must be between 0 and 1
- For example, can $p(t) = .6$ be an equilibrium price for period t?

.Other equilibria

- $p(t) = .6$ can have $p(t+1) = .1$ or $p(t+1) = .9$



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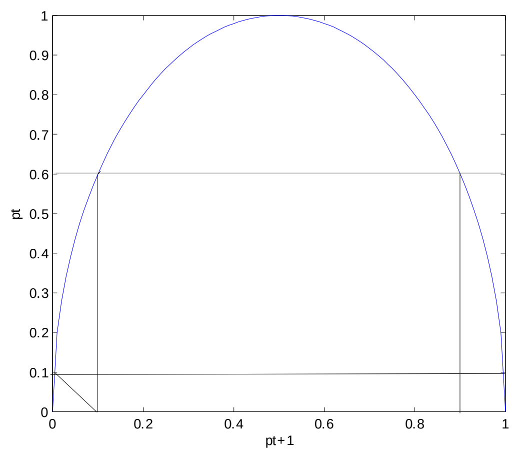

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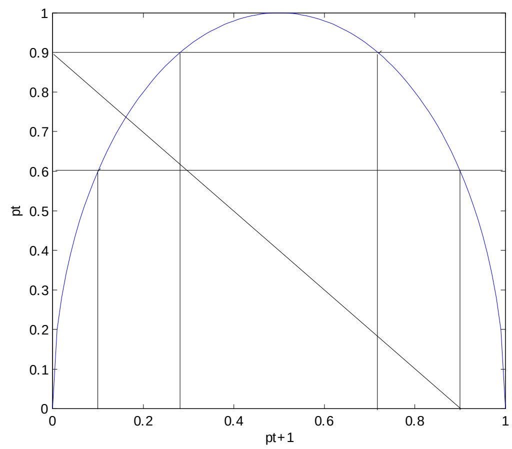


.Other equilibria

- $p(t+1) = .1$ can have $p(t+2) = 0.0025$ or $p(t+2) = 0.9975$

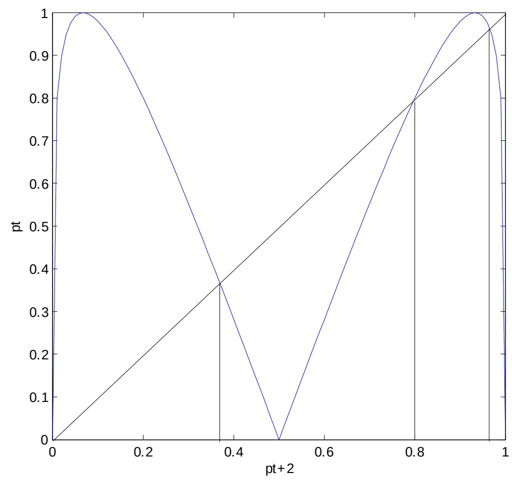
.Other equilibria

- $p(t+1) = .9$ can have $p(t+2) = 0.2821$ or $p(t+2) = 0.7179$

.Other equilibria

- One can repeat the process and find two $p(t+3)$ for each $p(t+2)$ and two $p(t+4)$ for each
- The number of possible equilibrium prices doubles for each period further in the future
- For $p(t) = .6$, there are 2^{20} different prices that can be equilibrium prices in periot $t + 20$.
- Notice how the price paths jump around
- NOTE: Every price in $p(t) \in [0, 1]$ is an equilibrium price
- This seems too much: Are there any "simple" paths
  - Is there a two period cycle possible?
  - Is there a three period cycle possible?

.Two period cycles



- The price function is

$$\begin{aligned} p(t) &= \left[4p\,(t+1) - 4p\,(t+1)^2\right]^{\frac{1}{2}} \\ &= G(p\,(t+1)) \end{aligned}$$

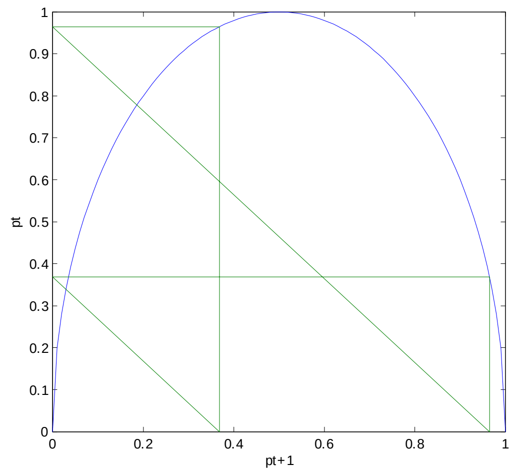
- Apply this price function twice to get

$$p(t) = G(G(p\,(t+2)))$$

- Look for fixed points: where $p(t) = p\,(t+2)$
- This price is a part of a two period cycle in the economy
- The graph on the next slide is the function $p(t) = G(G(p\,(t+2)))$
- The two cycle is $p\,(t) = 0.3685$ and $p\,(t+1) = 0.9648$
- Notice that the stationary states of .8 and 0 are also considered (uninteresting) 2 cycles

.Two period cycles
.How a two period cycle works
.Three period cycles

- Is it possible for the economy to have three period cycles?
- In these $p(t) = p(t+3)$ (and perhaps $p(t) \neq p(t+1) \neq p\,(t+2)$)



- Use the price function

$$\begin{aligned} p(t) &= \left[4p\,(t+1) - 4p\,(t+1)^2\right]^{\frac{1}{2}} \\ &= G(p\,(t+1)) \end{aligned}$$

- Apply two times

$$p(t) = G(G(G(p\,(t+3))))$$

- Look for a fixed point where $p(t) = p(t+3)$
- See following graph

.Three period cycles
.How the three cycle works
.Sharkovsky's ordering

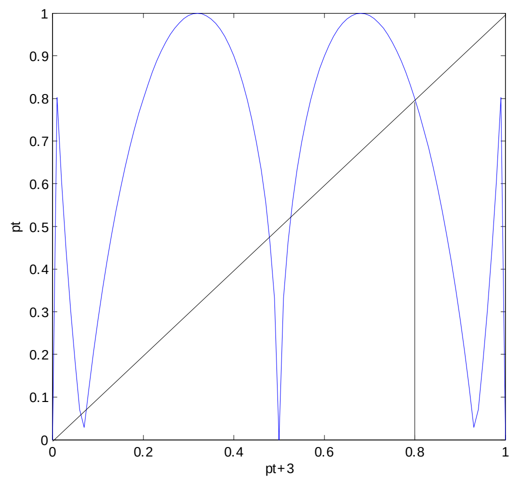
- Russian mathematician Oleksandr Mikolaiovich Sharkovsky invented an ordering

$$3, 5, 7, 9, ..., 2{\cdot}3, 2{\cdot}5, 2{\cdot}7, ..., 2^2{\cdot}3, 2^2{\cdot}5, ..., 2^4, 2^3, 2^2, 2, 1$$

- Sharkovsky's theorem

Theorem 1 *If a dynamical system has a cycle of order n, it also contains all cycles to the right of n in Sharkovsky's ordering*

.Equilibria with positive crops



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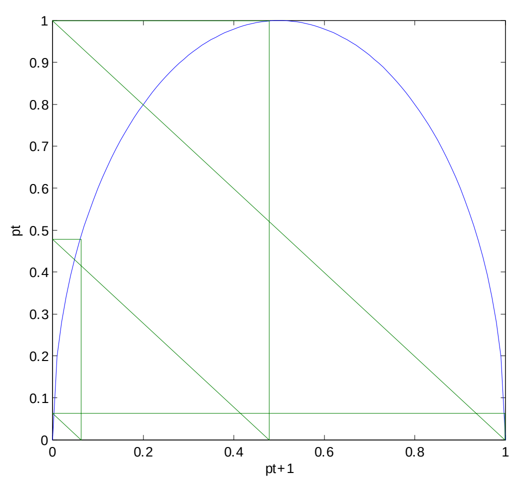

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pt
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0.4
0.6
0.8
1
pt+1

- What happens when $d(t) > 0$
- Crops show up in the no-arbitrage equation

$$p(t) = \frac{p(t+1) + d(t+1)}{r(t)}$$

- Putting crops back into the model result in the equilibrium equation of

$$p(t) = \frac{\beta \left[\frac{p(t+1)+d(t+1)}{p(t)}\right]}{1 + \beta \left[\frac{p(t+1)+d(t+1)}{p(t)}\right]^2}$$

- This equation simplifies to

$$p(t) = \left[\beta \left(p(t+1) + d(t+1)\right) - \beta \left(p(t+1) + d(t+1)\right)^2\right]^{\frac{1}{2}}$$

  where we have been using $\beta = 4$

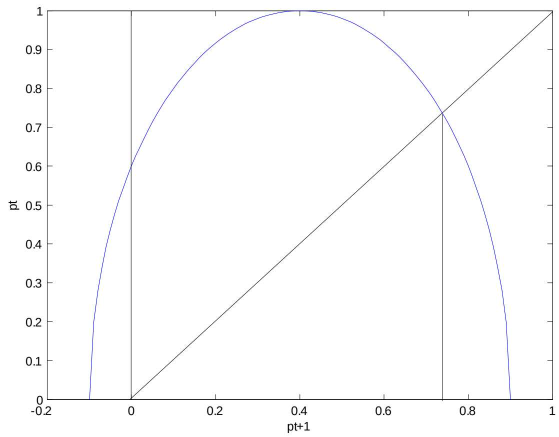
Equilibrium for small size crops

- Notice from

$$p(t) = \left[\beta \left(p(t+1) + d(t+1)\right) - \beta \left(p(t+1) + d(t+1)\right)^2\right]^{\frac{1}{2}}$$

  that a crop displaces the $p(t+1)$ to $p(t)$ curve to the left: to have the same value in the parenthesis as before, $p(t+1)$ is now smaller by the value of $d(t+1)$
- Multiple cycles are still possible if the crop is small enough
- Some time paths are no longer an equilibrium
  - prices cannot be negative so any path that demands a negative price for some data T can't be an equilibrium path
  - when $d(t) = .106922$, the three cycle has one point at zero: any bigger crop and three cycles are not possible
  - when $d(t) = .2$, the two cycle is zero and .8, so that for any crop bigger than .2, no two cycle exists

Equilibrium for small size crops
For large crop sizes, the economy converges to the stationary state



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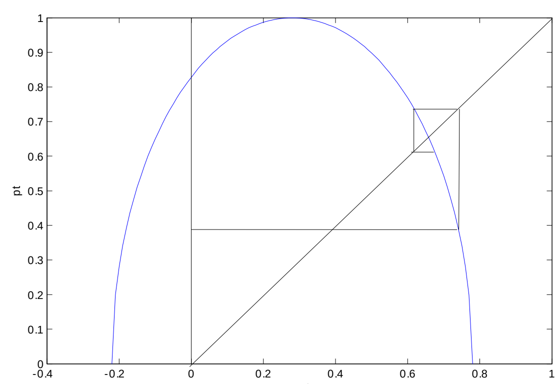

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