Ecoomic Growth
Class 10
More models of the AK type

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A variety of AK related models

1. Sobelow model: a combination of Solow and AK
2. Harrod Domar model: a very old model (1930's y 40's)
   - Has an AK type form in its production function
   - But this holds only some of the time
3. AK with optimizing consumers
   - use a CES utility function to get savings in an AK model
4. A simple physical capital - human capital model
   - where resources compete for producing human capital or physical capital

The Sobelow model: Combining Solow and AK

- Assume that the production function is

$$Y_t = AK_t + BK_t^{\theta} H_t^{1-\theta}$$

- This model gives constant returns to scale

$$A\lambda K_t + B\left(\lambda K_t\right)^{\theta}\left(\lambda H_t\right)^{1-\theta} = A\lambda K_t + \lambda BK_t^{\theta}H_t^{1-\theta} = \lambda Y_t$$

- Returns to capital and labor are positive

$$\frac{\partial Y_t}{\partial K_t} = A + \theta BK_t^{\theta-1}H_t^{1-\theta} > 0$$

$$\frac{\partial Y_t}{\partial H_t} = (1-\theta)\, BK_t^{\theta}H_t^{-\theta} > 0$$

- and are decreasing (check out the second derivatives)

The Sobelow model: Combining Solow and AK

- The model does not fulfill the Inada conditions (one of them)

$$\lim_{K\to\infty}\frac{\partial Y_t}{\partial K_t} = A \neq 0$$
$$\lim_{K\to 0}\frac{\partial Y_t}{\partial K_t} = \infty$$
$$\lim_{H\to\infty}\frac{\partial Y_t}{\partial H_t} = 0$$
$$\lim_{H\to 0}\frac{\partial Y_t}{\partial H_t} = \infty$$

- Fails in one of the conditions

The Sobelow model: Combining Solow and AK

- Find the per worker production function

$$y_t = Ak_t + Bk_t^{\theta}$$

- Use the law of motion of capital with the conditions for savings added

$$\begin{aligned}(1+n)\,k_{t+1} &= (1-\delta)\,k_t + sf(k_t)\\ &= (1-\delta)\,k_t + sAk_t + sBk_t^{\theta}\end{aligned}$$

- Write as the growth rate of capital

$$(1+n)\,\gamma_t = (1+n)\,\frac{(k_{t+1}-k_t)}{k_t} = sA + sBk_t^{\theta-1} - (n+\delta)$$

The Sobelow model: Combining Solow and AK
The Sobelow model: Combining Solow and AK

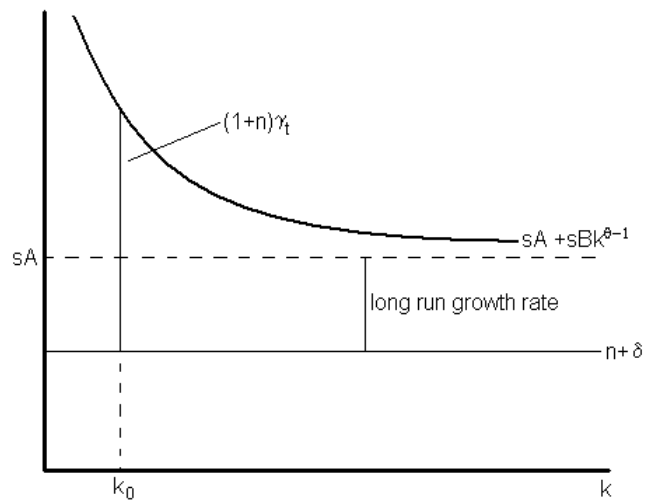
Harrod-Domar Model

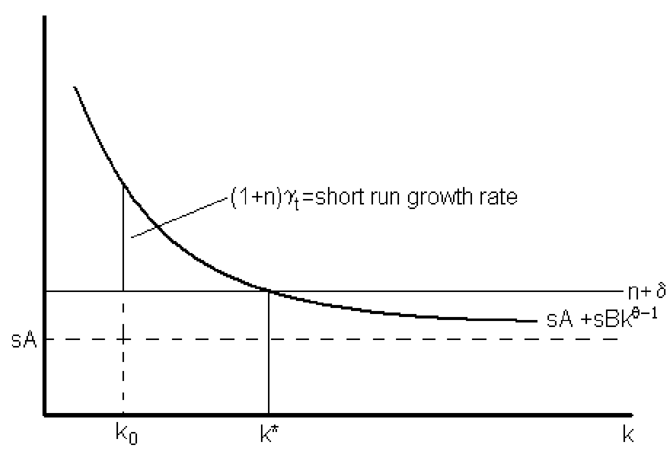
- Very early growth model
  - Harrod (1939), Domar (1946)
- Based on Leontief production functions

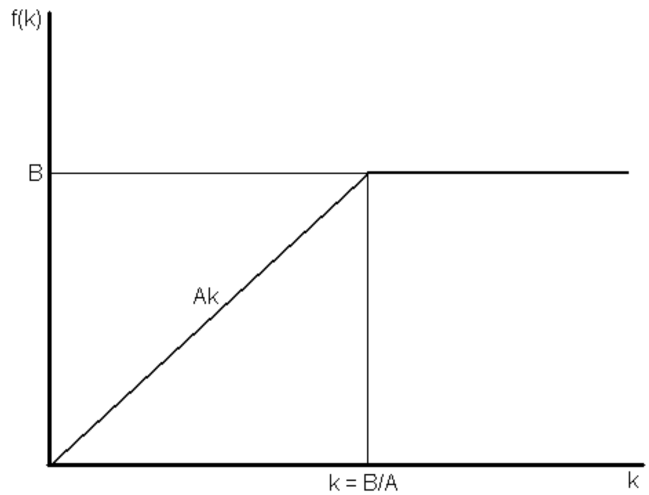
$$Y_t = \min\left[AK_t, BH_t\right]$$

- which in per worker terms is

$$y_t = \min\left[Ak_t, B\right]$$


(1+n)γt
sA +sBkθ−1
sA
long run growth rate
n+ δ
k0
k


(1+n)γt=short run growth rate
n+ δ
sA +sBkθ−1
sA
k0
k*
k



- this says that
  - if k_t is small enough so that $Ak_t < B$, then $y_t = Ak_t$
  - if k_t is big enough so that $Ak_t \geq B$, then $y_t = B$
- The dividing point is when $k_t = B/A$

Harrod-Domar Model (the production function)
Harrod-Domar Model

- The usual law of motion of capital is

$$(1+n)\,k_{t+1} = (1-\delta)\,k_t + sf(k_t)$$

- This now needs to be written as

$$(1+n)\,(k_{t+1} - k_t) = \begin{cases} sAk_t - (n+\delta)\,k_t & \text{if } \; k < B/A \\ sB - (n+\delta)\,k_t & \text{if } \; k \geq B/A \end{cases}$$

- or as

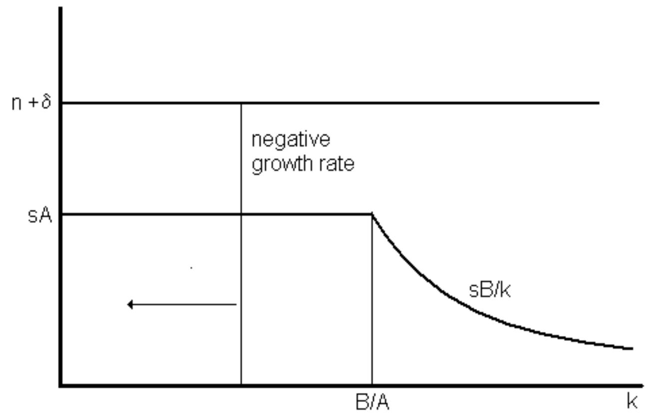
$$(1+n)\,\gamma_t = \begin{cases} sA - (n+\delta) & \text{if } \; k < B/A \\ sB/k_t - (n+\delta) & \text{if } \; k \geq B/A \end{cases}$$

Harrod-Domar Model (three possible equilibria: 1)

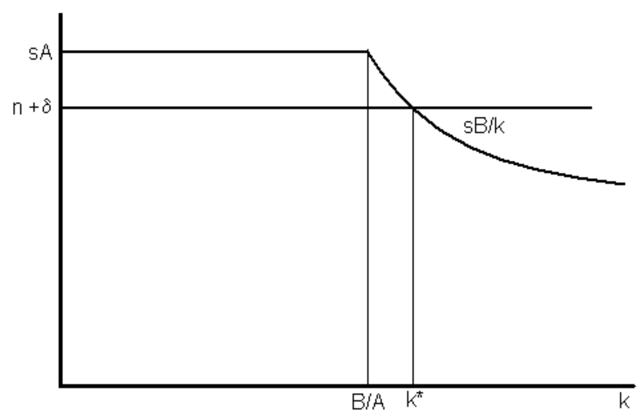
- equilibrium is at $k = 0$

Harrod-Domar Model (three possible equilibria: 2)

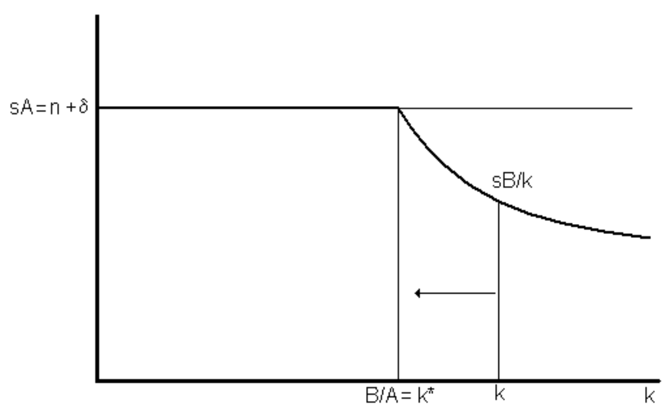
- stationary state at k^* but not all capital is utilized



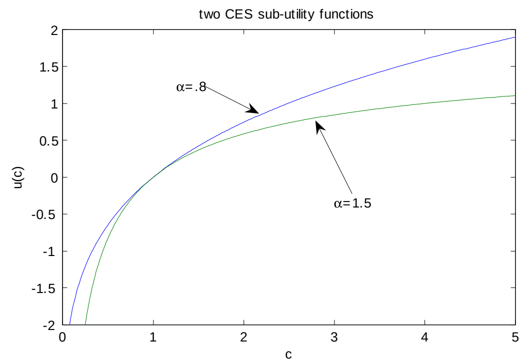
n + δ
negative
growth rate
sA
sB/k
B/A
k



sA
n + δ
sB/k
B/A
k*
k



sA = n + δ
sB/k
B/A = k*
k
k



Harrod-Domar Model (three possible equilibria: 3)

- Knife edge equilibrium

Harrod-Domar Model (possibility 3)

- If initial k is above B/A, the economy goes to $k^* = B/A$
- if initial k is below B/A, the economy stays there (no growth)
- Harrod and others at the University of Cambridge argued
  - the savings rate is not constant
  - with higher growth, the savings rate grows
  - workers have different savings rate = marginal propensity to save than capitalists
  - as economy grows, more income goes to the workers and the savings rate declines
  - think about what this means

CES sub-utility functions

- Constant elasticity of substitution sub-utility functions

$$u(c_t) = \frac{c_t^{1-\alpha} - 1}{1 - \alpha}$$

- α is the inverse of the elasticy of intertemporal substitution

CES sub-utility functions

- CES and log utility
$$\lim_{\alpha\to 1}\frac{c_t^{1-\alpha}-1}{1-\alpha}=\ln c_t$$
Note: if we just put in $\alpha=1$, we have
$$\frac{c_t^{1-\alpha}-1}{1-\alpha}=\frac{c_t^{1-1}-1}{1-1}=\frac{1-1}{0}=\frac{0}{0}$$

- l'Hopital's rule
$$\lim_{\alpha\to 1}\frac{c_t^{1-\alpha}-1}{1-\alpha}=\lim_{\alpha\to 1}\frac{\frac{d\left(c_t^{1-\alpha}-1\right)}{d\alpha}}{\frac{d(1-\alpha)}{d\alpha}}=\lim_{\alpha\to 1}\frac{-c_t^{1-\alpha}\ln c_t}{-1}=\ln c_t$$

First model: AK with optimization of consumers

- Households maximize
$$U_0=\sum_{t=0}^{\infty}\beta^t\frac{c_t^{1-\alpha}-1}{1-\alpha}$$

- with the budget constraints
$$k_{t+1}+c_t=y_t+(1-\delta)\,k_t$$

- and the production function
$$y_t=Ak_t$$

AK with optimization of consumers

- We will solve for constant growth paths (so we don't need the full policy function)
- Need first order conditions (necessary conditions)
- Use the Lagrangian
$$\mathcal{L}=\sum_{t=0}^{\infty}\beta^t\left[\frac{c_t^{1-\alpha}-1}{1-\alpha}+\lambda_t\left(k_{t+1}+c_t-Ak_t-(1-\delta)\,k_t\right)\right]$$

AK with optimization of consumers

- Fisrt order conditions are
$$\begin{aligned}\frac{\partial\mathcal{L}}{\partial c_s}&=c_s^{-\alpha}+\lambda_s=0\\ \frac{\partial\mathcal{L}}{\partial k_{s+1}}&=\lambda_s-\lambda_{s+1}\beta\left[A+(1-\delta)\right]=0\end{aligned}$$
so
$$\left(\frac{c_{s+1}}{c_s}\right)^{\alpha}=\beta\left[A+(1-\delta)\right].$$

AK with optimization of consumers

- Growth rate of consumption is (from the first order conditions)

$$\gamma_t^c = \left(\frac{c_{s+1}}{c_s} - 1\right) = \beta^{\frac{1}{\alpha}} \left[A + (1-\delta)\right]^{\frac{1}{\alpha}} - 1,$$

- Growth rate of capital we find from

$$k_{t+1} - k_t = Ak_t - \delta k_t - c_t$$

  and divide both sides by k_t to get

$$\gamma_t^k = \frac{k_{t+1} - k_t}{k_t} = A - \delta - \frac{c_t}{k_t}.$$

- Note that to have a constant growth rate of capital c_t/k_t needs to be constant

AK with optimization of consumers

- if c_t/k_t is a constant, then $\gamma_t^c = \gamma_t^k = \gamma_t^y = \gamma^*$
- Using this fact and combining the two growth rate equations, we get

$$\beta^{\frac{1}{\alpha}} \left[A + (1-\delta)\right]^{\frac{1}{\alpha}} - 1 = A - \delta - \frac{c}{k}$$

- In a stationary state growth path, c/k must be

$$\frac{c}{k} = A - \delta - \beta^{\frac{1}{\alpha}} \left[A + 1 - \delta\right]^{\frac{1}{\alpha}} + 1$$

- Everything on the right hand side are parameters, so we solve for c/k and then can find

$$\gamma^* = A - \delta - \frac{c}{k}.$$

AK with optimization of consumers (Example)

- Consider an economy with $A = 1$, $\alpha = 3$, $\delta = .1$, $\beta = .98$.
- Then

$$\frac{c}{k} = 1 - .1 - .98^{\frac{1}{3}} \left[1 + 1 - .1\right]^{\frac{1}{3}} + 1$$

- $c/k = 0.66975$
- and

$$\gamma^* = 1 - .1 - 0.66975 = 0.23025$$

- So the steady state growth rate is 23.025%

Finding the savings rate

- The budget constraint give us

$$s_t = y_t - c_t$$

and the saving rate (as a fraction of output) is

$$\frac{s_t}{y_t} = 1 - \frac{c_t}{y_t}.$$

- The production function is

$$y_t = Ak_t$$

we we can write

$$\widehat{s} = \frac{s_t}{y_t} = 1 - \frac{c_t}{y_t} = 1 - \frac{c_t}{Ak_t} = 1 - \frac{1}{1}0.66975 = 0.33025$$

AK with optimization of consumers

- Need to check if the utility function is well defined
- If

$$U_0 = \sum_{t=0}^{\infty} \beta^t \frac{c_t^{1-\alpha} - 1}{1-\alpha} = \infty$$

we have a problem

- Need to find the conditions under which this has a finite value

AK with optimization of consumers

- Given a constant growth rate of consumption, $\gamma_t^c = \gamma^*$, we can write

$$\begin{aligned} c_t &= (1+\gamma_t^c)^t c_0 \\ &= \beta^{\frac{t}{\alpha}} \left[A + (1-\delta)\right]^{\frac{t}{\alpha}} c_0 \end{aligned}$$

- From

$$\begin{aligned} \sum_{t=0}^{\infty} \beta^t \frac{c_t^{1-\alpha} - 1}{1-\alpha} &= \sum_{t=0}^{\infty} \beta^t \frac{c_t^{1-\alpha}}{1-\alpha} - \frac{1}{1-\alpha} \sum_{t=0}^{\infty} \beta^t \\ &= \frac{1}{1-\alpha} \sum_{t=0}^{\infty} \beta^t c_t^{1-\alpha} - \frac{1}{(1-\alpha)(1-\beta)} \end{aligned}$$

- we need

$$\sum_{t=0}^{\infty} \beta^t c_t^{1-\alpha} < \infty$$

AK with optimization of consumers

- Substituting in our calculation for consumption, we have

$$\sum_{t=0}^{\infty} \beta^t \left(\beta^{\frac{t}{\alpha}} \left[A + (1-\delta)\right]^{\frac{t}{\alpha}} c_0\right)^{1-\alpha} = c_0^{1-\alpha} \sum_{t=0}^{\infty} \left(\beta \left(\beta \left[A + (1-\delta)\right]\right)^{\frac{(1-\alpha)}{\alpha}}\right)^t$$

- and we need

$$\beta \left(\beta \left[A + (1-\delta)\right]\right)^{\frac{(1-\alpha)}{\alpha}} < 1$$

- which occurs when

$$\left[A + (1-\delta)\right]^{(1-\alpha)} < \beta^{-1}$$

- Note that for our example we are using $\alpha = 3$, so the condition for the example is

$$\frac{1}{\left[A + (1-\delta)\right]^2} < \frac{1}{\beta}$$

AK with optimization of consumers

- Result:
- We can find an equilibrium with optimizing consumers in a model with AK production function
- We find the constant growth rate path
- There, all variables grow at the same rate
- Need to check that the transversality condition holds and therefore that utility is difined

Second model: AK models as models of human capital

- We can consider the conditions under which a model with production that uses human capital can become an AK type model
- Consider the production function

$$Y_t = BK_t^{\theta} H_t^{1-\theta}$$

- and assume that both one unit of capital and one unit of human capital can be produced with one unit of the good, so the budget constraint can be written as (we assume that depreciation is the same for both types of capital)

$$K_{t+1} + H_{t+1} = w_t H_t + r_t K_t + (1-\delta) H_t + (1-\delta) K_t.$$

AK models as models of human capital

- The return on a unit of human capital is $w_t - \delta$
- The return on a unit of physical capital is $r_t - \delta$
- In equilibrium, these returns must be equal (since the cost of a unit of each is the same)
- From competitive factor markets, we have

$$w_t = (1-\theta)\, BK_t^{\theta} H_t^{-\theta} = (1-\theta)\, \frac{Y_t}{H_t}$$

and

$$r_t = \theta B K_t^{\theta-1} H_t^{1-\theta} = \theta \frac{Y_t}{K_t}.$$

AK models as models of human capital

- Since depreciation is the same for both physical and human capital, the equilibrium condition is

$$w_t = r_t$$

- This can be written as

$$(1-\theta)\, \frac{Y_t}{H_t} = \theta \frac{Y_t}{K_t}$$

- which can be simplified to get

$$H_t = \frac{(1-\theta)}{\theta} K_t.$$

AK models as models of human capital

- Putting this last result into the production function, we get that in equilibrium

$$\begin{aligned} Y_t &= BK_t^{\theta} H_t^{1-\theta} = BK_t^{\theta} \left(\frac{1-\theta}{\theta} K_t \right)^{1-\theta} \\ &= B \left(\frac{1-\theta}{\theta} \right)^{1-\theta} K_t \\ &= AK_t \end{aligned}$$

where the constant A is simply

$$A = B \left(\frac{1-\theta}{\theta} \right)^{1-\theta}.$$

AK models as models of human capital

- Result:

- with same depreciation rates and same cost of production,
- a model of production with production function

$$Y_t = BK_t^{\theta} H_t^{1-\theta}$$

- becomes a model with production

$$Y_t = AK_t$$

- where

$$A = B\left(\frac{1-\theta}{\theta}\right)^{1-\theta}.$$

New topic

Introduction to Technological Change

Concepts of Technology

- Types of technological change
- **Process**
  - produce higher quality versions of existing products
  - or produce an existing product at lower cost
- **Product**
  - New products are added to the basket of goods
- Nature of technological change: macro or micro
  - **Macro** changes are radical innovations (relatively rare)
    * introduction of general purpose technologies
    * such as steam engine, electricity, computer
  - **Micro** changes are smaller (the vast majority)
    * newer versions of existing items
    * better processes of production of existing product
    * reducing costs of production

Some characteristics of technology

- Nonrivalry
  - one producer improving efficiency doesn't prevent others from doing the same

- potentially, all produces can increase efficiency
- can be difficult to prevent others from using

- Nonrivalry can be like a externality
  - think of the production function

$$Y = f(K, L, A)$$

  - can have constant returns to scale in K and L
  - and **increasing returns to scale** in K, L, and A

- Not a pure public goods
  - may be excludable
  - especially if there are patents (government makes them excludable)
  - but may be through industral secrets

Invention and profits

- Why do people make inventions (inovations)?
- Science: search for basic knowledge
  - frequently subsidized: new knowledge not immediately exploitable
- Potential profits for invention (which may use scientific knowledge)
  - reward from British navy for method to determine longitude (a clock, finally)
  - search to solve a costly problem
    * removing seeds from cotton
    * pump the water out of mines
  - potential market size matters
    * although frequently underestimated
    * example of estimates for number of computers needed in USA

Simple model of technology (partial equilibrium)

- Marginal cost of production of one unit of the good = ψ
- Demand curve of the industry is

$$Q = D(p)$$

- where Q is the quantity sold and p is the price

- Assume $D(\psi) > 0$, so there is postive demand when price = marginal cost
- Elasticity of demand is

$$\varepsilon_D(p) = -\frac{pD'(p)}{D(p)} \in (1, \infty)$$

- this last implies that $\varepsilon_D(p) > 1$, so there exists a well defined monopolist price
- Assume that an innovation costs $\mu > 0$ and reduces marginal costs by $1/\lambda$ where $\lambda > 1$
- Marginal costs after an innovation are ψ/λ

Perfect competition

- There are N firms in the industry with access to the existing technology
- In initial equilibrium $p = \psi$ and the profits of a firm are

$$\pi_i = (p - \psi) Q_i = 0$$

  - If all firms are the same, $Q_i = D(\psi)/N$
- What are the incentives of a firm to spend μ to make an innovation

Perfect competition

- Suppose that firm 1 spends for the innovation
  - Assume the innovation is nonexcludable (so all firms can use it)
  - Profits to firm 1 are

$$\pi_1^* = (p^* - \psi/\lambda) Q_1^* - \mu$$

  - But all firms can use the innovation, so $p^* = \psi/\lambda$, and

$$\pi_1^* = -\mu$$

    and

$$\pi_{i\neq 1}^* = 0$$

- There are no incentives for any firm to innovate: no innovation occurs

Ex post monopoly power on the innovation

- Suppose that the inventing firm can prevent other firms from using its innovation

- Suppose that the innovation is **drastic** (λ is large) and the firm becomes a monopolist
- Firm maximizes (wrt p)

$$\pi_1' = (p - \psi/\lambda)\, D(p) - \mu$$

Ex post monopoly power on the innovation

- First order condition is

$$0 = D(p) + (p - \psi/\lambda)\, D'(p)$$

or

$$\frac{D(p)}{pD'(p)} + 1 = \frac{\psi/\lambda}{p}$$

or

$$p^m = \frac{\psi/\lambda}{1 - \varepsilon(p^m)^{-1}}$$

- If $p^m < \psi$, this firm will charge the price p^m and will take over the whole market

Ex post monopoly power on the innovation

- If If $p^m > \psi$ the innovation is NOT drastic (Limit pricing solution)
- The innovating firm cannot charge p^m, since the other firms charge $p^* = \psi$
- The innovating firm will have profits of

$$\pi_1' = (p^* - \psi/\lambda)\, \frac{D(\psi)}{N} - \mu$$

- The other firms will have zero profits
- Of course, other firms have incentives to try and steal the technology
  - stealing may be cheaper than innovating