An endogenous growth model with human capital and learning

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One can get an AK model by directly introducing human capital accumulation. The model presented here is a simplified version of that of Robert Lucas (1988)[1]. Lucas consider a learning rule where there are no diminishing returns on the stock of human capital in the production of new human capital. In other words, human capital accumulates according to a learning rule,

$$h_{t+1} = dh_t^{\phi} (1 - u),$$

where $\phi = 1$. In the learning rule, $1 - u$ is the fraction of non-leisure time that each worker spends learning and d is a parameter that indicates the relative efficiency of learning. Lucas argues that learning has the property of non-rivalry, that I can learn at the same time as you are learning. In fact, that is why learning frequently takes place in groups (such as a class). Non-rivalry in learning helps make the assumption that $\phi = 1$ seem reasonable.

0.1 Goods production

Suppose that all workers work u fraction and study $1 - u$ fraction of their non-leisure time. Each worker earns income as an "effective" worker who, with h units of accumulated human capital, is equivalent to h basic workers. If there are N workers and each of them has h units of human capital, then the effective labor supply is equal to uhN. A worker with h units of human capital earns a salary of $w_t h$ in period t, where w_t is the period t salary that is earned by each unit of effective labor (it is equal to the marginal product of a unit of effective labor). The Cobb-Douglas style production function is

$$Y_t = AK_t^{\theta} (uh_t N_t)^{1-\theta},$$

where A is the constant level of technology. This production function can be written in per worker terms (there are N_t workers) as

$$y_t = \frac{Y_t}{N_t} = \frac{AK_t^{\theta} (uh_t N_t)^{1-\theta} h_t^{\eta}}{N_t} = Ak_t^{\theta} (uh_t)^{1-\theta}$$

[1] Robert E. Lucas, Jr. (1988), "On the Mechanics of Economic Development", *Journal of Monetary Economics*, 22, pp. 1 - 42.

Following the basic Solow model, we use the per capita physical capital accumulation rule,

$$k_{t+1} = (1-\delta)\,k_t + s y_t,$$

where s is a constant savings rate.

0.2 Learning and human capital production

Human capital accumulation occurs according to the learning rule

$$h_{t+1} = d h_t\,(1-u)\,.$$

The idea here is that workers learn based on the amount of time they spend learning, $(1-u)$, and on the amount of per capita knowledge that there is in the economy, h_t. If there is more knowledge, learning is easier. The parameter d indicates how efficient learning is. Clearly we want to consider economies where $d\,(1-u) > 1$ so that we have positive net growth in knowledge. The gross growth rate of human capital is equal to

$$\gamma^h = \frac{h_{t+1}}{h_t} = \frac{d h_t\,(1-u)}{h_t} = d\,(1-u)\,.$$

There is no time subscript on the growth rate of human capital because it is constant.

0.3 Solving for growth rates

Using the equation for capital accumulation, we find the gross growth rate of capital is equal to

$$\begin{aligned}
\gamma_t^k &= \frac{k_{t+1}}{k_t} = (1-\delta) + s\frac{y_t}{k_t}, \\
&= (1-\delta) + s\frac{A k_t^{\theta}\,(u h_t)^{1-\theta}}{k_t}, \\
&= (1-\delta) + s A u^{1-\theta}\left(\frac{h_t}{k_t}\right)^{1-\theta}.
\end{aligned}$$

The growth rate of physical capital depends only on the ratio of human capital to physical capital.

Using the production function, one can calculate the gross growth rate of output as

$$\begin{aligned}
\gamma_t^y &= \frac{y_{t+1}}{y_t} = \frac{A k_{t+1}^{\theta}\,(u h_{t+1})^{1-\theta}}{A k_t^{\theta}\,(u h_t)^{1-\theta}} = \left(\frac{k_{t+1}}{k_t}\right)^{\theta}\left(\frac{h_{t+1}}{h_t}\right)^{1-\theta} \\
&= \left(\frac{k_{t+1}}{k_t}\right)^{\theta}(d\,(1-u))^{1-\theta} \\
&= \left(\gamma_t^k\right)^{\theta}(d\,(1-u))^{1-\theta}
\end{aligned}$$

From the production function we find that the growth rate of output is a weighed average of the growth rates of physical capital and of human capital.

0.4 The balanced growth path

A balanced growth path is one where the growth rate of output is constant. From the equation for the growth rate of output, one can deduce that output growth will be constant (on a balanced growth path) only if the growth rate of the physical capital, γ_t^k, is a constant. The equation for the rate of growth of the capital stock is

$$\gamma_t^k = (1-\delta) + sAu^{1-\theta}\left(\frac{h_t}{k_t}\right)^{1-\theta},$$

and this growth rate will be a constant only when the human capital-physical capital ratio is constant. It is quite direct to see that the human capital-physical ratio is constant only when the rate of growth of capital is equal to the rate of growth of human capital or when $\gamma_t^k = d(1-u)$. Setting the growth rate of physical capital equal to $d(1-u)$ in the above equation, we find that the human capital-capital ratio along a balanced growth path must be

$$\overline{h/k} = \left[\frac{d(1-u)-(1-\delta)}{sAu^{1-\theta}}\right]^{\frac{1}{1-\theta}}$$

Along the path with $h_t/k_t = \overline{h/k}$, the growth rate of output is

$$\gamma_t^y = \left(\gamma_t^k\right)^\theta (d(1-u))^{1-\theta} = (d(1-u))^\theta (d(1-u))^{1-\theta} = d(1-u).$$

Along the balanced growth path, human capital, physical capital, and output all grow at the same rate.

Figure 1 shows the balanced growth paths for four economies. All the economies have parameters of $A = 1$, $\theta = .4$, $\delta = .1$, $s = .15$, $d = 2.1$, and an initial human capital stock of $h_1 = 4$. The figure shows the balanced growth paths for different values of the fraction of time spent working, u, with values of $u = .5$, .51, .52, and .53. Notice that for higher fractions of time worked, a higher amount of capital and a higher output is associated with the initial 4 units of human capital. However, increasing the fraction of time worked dramatically reduces human capital growth and this results in very different time paths. By period 20, the economy where people have been studying more, $u = .50$, is superior to all the others. Notice that the path of the model is quite sensitive to the value of u and relatively small changes can result in very different time paths.

0.5 Dynamics

The next question is whether the balanced growth path matters or not. For it to matter, we would like the economies that are not on the balanced growth path

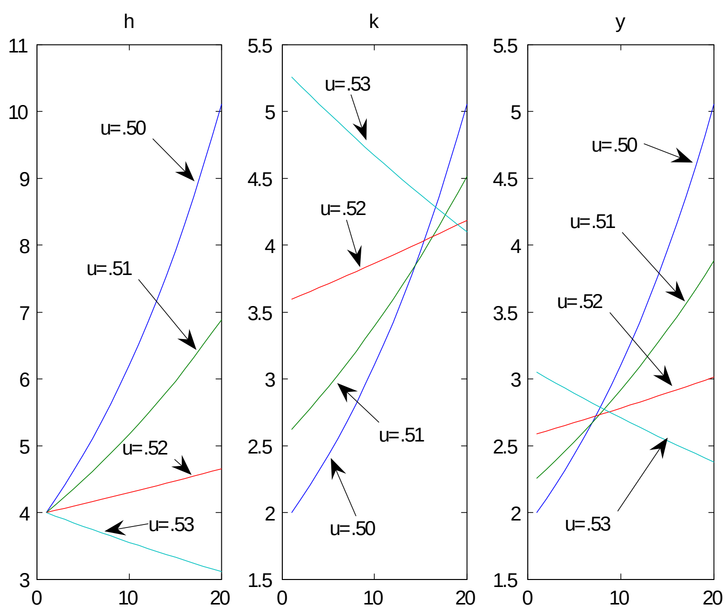

Figure 1: Four balanced growth paths

to converge towards it. Since an economy with the human capital - physical capital ratio of $\overline{h/k}$ will be on the balanced growth path, we want to consider what happens to economies with $h_t/k_t < \overline{h/k}$ or $h_t/k_t > \overline{h/k}$.

Suppose that there is more capital than is required to get the right ratio, assume that $h_t/k_t < \overline{h/k}$. In this case, the growth rate is

$$\gamma_t^k = (1-\delta) + sAu^{1-\theta}\left(\frac{h_t}{k_t}\right)^{1-\theta} < (1-\delta) + sAu^{1-\theta}\left(h/k^*\right)^{1-\theta}.$$

In this case, the economy begins with more physical than is required to be on the balanced growth path and this capital stock grows slower than human capital until the balanced growth path ratio of human capital to physical capital is obtained.

Assume that $h_t/k_t > \overline{h/k}$, and there is less physical capital than is required to have the balanced growth path ratio of human capital to physical. In that case, the growth rate of physical capital is

$$\gamma_t^k = (1-\delta) + sAu^{1-\theta}\left(\frac{h_t}{k_t}\right)^{1-\theta} > (1-\delta) + sAu^{1-\theta}\left(h/k^*\right)^{1-\theta}.$$

The physical capital stock grows faster than human capital until the ratio of the two is equal to h/k^*. The economy is then on the balanced growth path.

0.6 Why is this model an AK model

To see why this model has the characteristics of an AK model, we look at the production function,

$$Y_t = AK_t^{\theta}\left(uh_tN_t\right)^{1-\theta}.$$

Along the balanced growth path, the physical capital - human capital ratio is constant. With constant population, one can write the production function as

$$Y_t = A\left(\frac{Nk_t}{uh_tN}\right)^{\theta} uNh_t.$$

Since human capital grows at the same rate as physical capital, the parameter

$$\overline{A} = A\left(\frac{k_t}{uh_t}\right)^{\theta} uN = A\left(u\overline{h/k}\right)^{-\theta} uN$$

is a constant along the balanced growth path. Then the production function can be written as

$$Y_t = \overline{A}h_t.$$

The model has an Ah production function, which is the same form as an AK production function.

0.7 Some characteristics of the model

One can imagine that different countries have different savings rates, s, and different rates of spending their non-leisure time on working or on learning, u. The one that has the most interesting effects is the fraction of time spent working so we will do that first.

0.7.1 Different fraction of time working

Suppose that one country has a u that is higher than others. People in this country work more and study less. Since they study less, the rate of growth of human capital will be smaller. Recall that the function for human capital accumulation is

$$h_{t+1} = dh_t \left(1 - u\right)$$

and that this implies that the growth rate of human capital is

$$\gamma^h = \frac{h_{t+1}}{h_t} = d\left(1 - u\right).$$

A higher u means that $1 - u$ is smaller and that γ^h is smaller. Once $1 - u$ gets as small as $1/d$, net human capital accumulation become zero. We showed that along a balanced growth path, the growth rate of the economy (output) was equal to the growth rate of human capital, so that if the fraction of time worked, u, gets as large as $1 - 1/d$, the economy doesn't grow.

A high u has an interesting effect on the balanced growth path ratio of human capital to physical capital and, therefore, to output. The equation for determining the balanced growth $\overline{h/k}$ is

$$\overline{h/k} = \left[\frac{d\left(1-u\right) - \left(1-\delta\right)}{sAu^{1-\theta}}\right]^{\frac{1}{1-\theta}}.$$

The derivative of $\overline{h/k}$ with respect to u is

$$\frac{\partial \overline{h/k}}{\partial u} = -\frac{1}{1-\theta}\left[\frac{d\left(1-u\right) - \left(1-\delta\right)}{sAu^{1-\theta}}\right]^{\frac{\theta}{1-\theta}}\left[\frac{ud + \left(1-\theta\right)\left[d\left(1-u\right) - \left(1-\delta\right)\right]}{sAu^{2-\theta}}\right]$$

which is negative when $d\left(1-u\right) > \left(1-\delta\right)$. Note that when $d\left(1-u\right) < \left(1-\delta\right)$, no balanced growth path exists since either the $1/(1-\theta)$ power does not exist or it is negative (and there is no sense to a negative $\overline{h/k}$). The negative derivative implies that for balanced growth paths with higher u, $\overline{h/k}$ is lower and there is more physical capital per unit of human capital. Two economies with the same amount of per capita human capital, the one with the higher u will have higher per capita output. This can be seen from the per capita version of the production function,

$$y_t = Ak_t^{\theta}\left(uh_t\right)^{1-\theta},$$

where a higher u and a higher k_t imply a higher y_t.

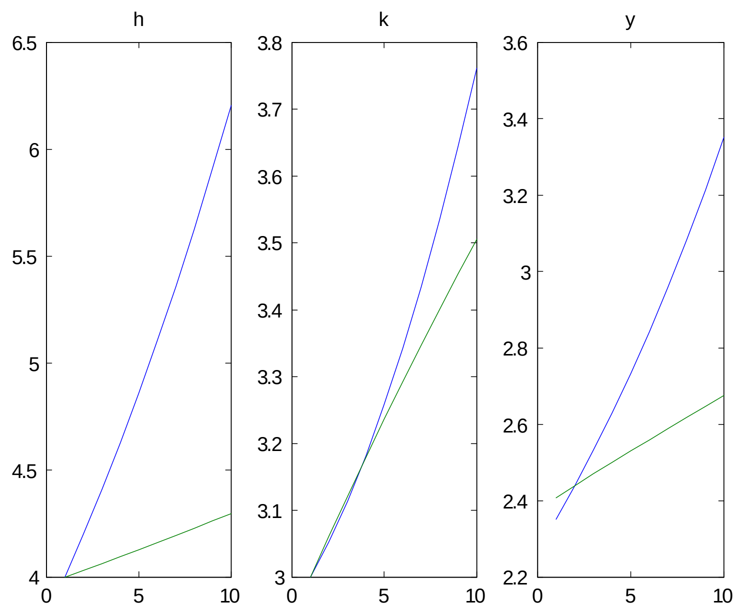

Figure 2: Two economies with different fraction of time spent working

Figure 2 shows a pair of countries using the parameters $A = 1$, $\theta = .4, \delta = .1, s = .15, d = 2.1$, and $u = .5$ in one case and $u = .52$ in the other case. Both economies begin with $k_1 = 3$ and $h_1 = 4$ and are run for 10 periods. As the h graph shows, human capital grows slower in the economy with $u = .52$. What is interesting is what happens to physical capital and to output. Output is initially higher in the economy with the higher u but is quickly passed by the economy with the higher growth rate. Physical capital is more complex with the economy with the higher u having higher physical capital for a number of periods (living off the initial boost) and then falling behind. Note that the graphs in Figure 2 are not balanced growth paths (those are shown in Figure 1) but are time paths for economies with different u's that begin with the same amount of human and physical capital.

The issue discussed in this section is important because in very poor countries, young people are frequently pulled out of school and put to work by and for the family. In those countries, u is higher, and while that might make the short term output higher, especially for the family, it has a negative effect on the rate of economic growth. This effect might be large enough to generate a "poverty trap" in which the aggregate economy does not grow.

0.7.2 Different savings rates

Suppose that we have two countries with different savings rates, but are otherwise equal. They will have the same growth rates along their balanced growth paths since they both are accumulating human capital at the same rates (they have identical learning functions). From the balanced growth $\overline{h/k}$,

$$\overline{h/k} = \left[\frac{d\left(1-u\right)-\left(1-\delta\right)}{sAu^{1-\theta}}\right]^{\frac{1}{1-\theta}},$$

we can easily see that a higher s implies that, along the balanced growth path, for each level of human capital, physical capital will be higher. A larger s will produce a smaller $\overline{h/k}$ and this implies a larger k_t for each h_t. Since there is more physical capital at each level of human capital, from

$$y_t = Ak_t^{\theta}\left(uh_t\right)^{1-\theta},$$

we can see that output per capita will also be higher.

Note: the growth rates of the two countries are the same along their balanced growth paths, but the one with the higher savings rate will have more output. Since each economy converges to it balanced growth path and both have the same amount of human capital, if they begin with the same amounts of human and physical capital, the economy with the higher savings rate will grow faster since it will be accumulating physical capital faster.

Figure 3 shows the same two economies as in Figure 2 except that the fraction of time working is the same for both, $u = .5$ and initial capital for both countries was set to $k_1 = 5.$, Here the faster growing one has a savings rate of .15 and the slower growing one has a savings rate of , 14. Note that human capital grows at exactly the same rate in both economies but that the higher savings rate means greater physical capital accumulation and higher output. The higher value for initial per capita physical capital was chosen to show how both the economies converge to their respective balanced growth path.

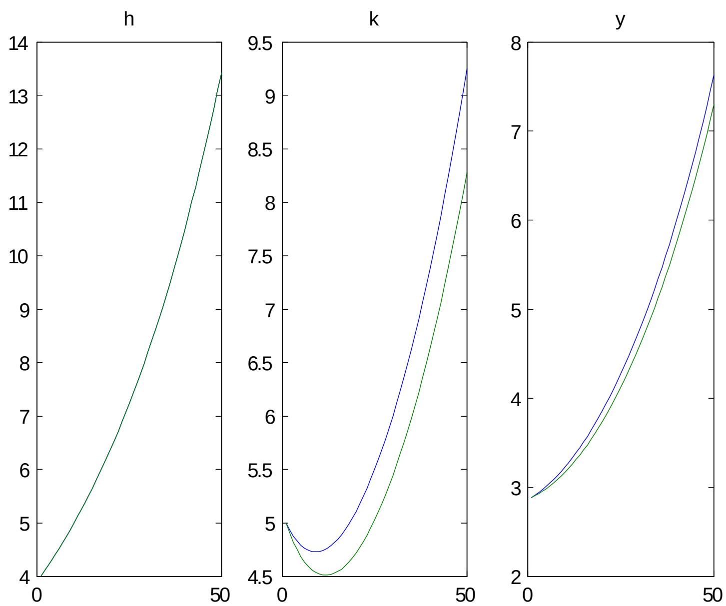

Figure 3: Comparing economies with different savings rates