1 Optimizing in an AK model

Suppose that

$$U_0 = \sum_{t=0}^{\infty} \beta^t u\left(c_t\right)$$

with the budget constraints

$$k_{t+1} + c_t = y_t + \left(1-\delta\right) k_t$$

and

$$y_t = Ak_t$$

Suppose that we use the CES sub-utility function

$$u\left(c_t\right) = \frac{c_t^{1-\alpha} - 1}{1-\alpha}.$$

Here, α is the inverse of the elasticity of intertemporal substitution. The Lagrangean can be written as

$$\mathcal{L} = \sum_{t=0}^{\infty} \beta^t \left[\frac{c_t^{1-\alpha} - 1}{1-\alpha} + \lambda_t \left(k_{t+1} + c_t - Ak_t - \left(1-\delta\right) k_t\right) \right]$$

and the first order conditions are

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial c_s} &= c_s^{-\alpha} + \lambda_s = 0 \\ \frac{\partial \mathcal{L}}{\partial k_{s+1}} &= \lambda_s - \lambda_{s+1} \beta \left[A + \left(1-\delta\right)\right] = 0 \end{aligned}$$

so

$$\left(\frac{c_{s+1}}{c_s} \right)^{\alpha} = \beta \left[A + \left(1-\delta\right)\right].$$

This implies that

$$\gamma_t^c = \left(\frac{c_{s+1}}{c_s} - 1 \right) = \beta^{\frac{1}{\alpha}} \left[A + \left(1-\delta\right)\right]^{\frac{1}{\alpha}} - 1,$$

where γ_t^c is the growth rate of consumption. In equilibrium, consumption growth is constant and is lower as α is larger, is larger as β is larger.

To find the equilibrium growth rate of capital, write the budget constraint as

$$k_{t+1} - k_t = Ak_t - \delta k_t - c_t$$

and divide both sides by k_t to get

$$\gamma_t^k = \frac{k_{t+1} - k_t}{k_t} = A - \delta - \frac{c_t}{k_t}.$$

To consider a constant growth rate economy, $\gamma_t^k = \overline{\gamma}^k$, the ratio of consumption to capital must be a constant, the part $\frac{c_t}{k_t}$ of the above equation must be a constant. For that to hold, it must be the case that $\gamma_t^k = \gamma_t^c$, and since $y_t = Ak_t$, $\gamma_t^y = \gamma_t^k = \gamma_t^c$.

Combining the two results implies that

$$\beta^{\frac{1}{\alpha}} \left[A + (1-\delta)\right]^{\frac{1}{\alpha}} - 1 = A - \delta - \frac{c_t}{k_t}$$

and in a stationary state growth path the ratio between consumption and capital is

$$\frac{c_t}{k_t} = A - \delta - \beta^{\frac{1}{\alpha}} \left[A + (1-\delta)\right]^{\frac{1}{\alpha}} + 1$$

The utility function is

$$U_0 = \sum_{t=0}^{\infty} \beta^t \frac{c_t^{1-\alpha} - 1}{1-\alpha}$$

and we need to consider under what conditions this has a finite value when consumption grows at the constant rate

$$\gamma_t^c = \beta^{\frac{1}{\alpha}} \left[A + (1-\delta)\right]^{\frac{1}{\alpha}} - 1,$$

If it grows at this constant rate, we can write c_t as

$$\begin{aligned} c_t &= \left(1 + \gamma_t^c\right)^t c_0 \\ &= \left(\beta \left[A + (1-\delta)\right]\right)^{\frac{t}{\alpha}} c_0. \end{aligned}$$

For the utility function to be finite, we need

$$\begin{aligned} \sum_{t=0}^{\infty} \beta^t \frac{c_t^{1-\alpha} - 1}{1-\alpha} &= \sum_{t=0}^{\infty} \beta^t \frac{c_t^{1-\alpha}}{1-\alpha} - \frac{1}{1-\alpha} \sum_{t=0}^{\infty} \beta^t \\ &= \frac{1}{1-\alpha} \sum_{t=0}^{\infty} \beta^t c_t^{1-\alpha} - \frac{1}{(1-\alpha)(1-\beta)} \end{aligned}$$

to be finite and this means that

$$\sum_{t=0}^{\infty} \beta^t c_t^{1-\alpha}$$

must be finite. Replacing c_t from above, this gives

$$\sum_{t=0}^{\infty} \beta^t \left(\left(1 + \gamma_t^c\right)^t c_0\right)^{1-\alpha} = c_0^{1-\alpha} \sum_{t=0}^{\infty} \left(\beta \left(\beta \left[A + (1-\delta)\right]\right)^{\left(\frac{1-\alpha}{\alpha}\right)}\right)^t$$

so we need that

$$\beta \left(\beta \left[A + (1-\delta)\right]\right)^{\left(\frac{1-\alpha}{\alpha}\right)} < 1.$$

This gives

$$\left[A + (1-\delta)\right]^{1-\alpha} < \beta^{-1}$$

as the condition we need to be sure of a finite value for the discounted utility.